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HISTORY

*van der Waerden, B. L. **Ontwakende Wetenschap. Egyptische, Babylonische en Griekse Wiskunde. [Awakening Science. Egyptian, Babylonian and Greek Mathematics].** P. Noordhoff, Groningen, 1950. 332 pp. 13.50 florins.

This is the first book which bases a full discussion of Greek mathematics on a solid discussion of pre-Greek mathematics. Carefully using the best sources available at present, the author acquaints the reader not only with the work of Neugebauer and Heath, but also with that of the philological critics who centered around the "Quellen und Studien." He thus aims at certain main goals, which he describes as follows: (a) to show how Thales and Pythagoras started from Babylonian mathematics, but gave it an entirely different specifically Greek character; (b) how mathematics inside and outside of the school of the Pythagoreans was gradually developed, satisfying increasingly the demands of logic; and (c) how through Plato's friends Theaitetos and Eudoxus that degree of perfection, beauty and rigor was reached which we admire in Euclid's Elements. Chapter I (pp. 15–42) deals with the Egyptians, chapter II (pp. 43–69) with the number systems, number notation and arithmetic of the Sumerians, the Greeks and the Indians, chapter III (pp. 70–91) with the Babylonians, chapter IV (pp. 92–118) with the age of Thales and Pythagoras, chapter V (pp. 119–164) with the Golden Age, chapter VI (pp. 165–223) with the age of Plato (this includes Euclid), chapter VII (pp. 224–291) with the Alexandrinians (notably Archimedes, Eratosthenes and Apollonios), while chapter VIII (pp. 292–321) deals with the decline of Greek mathematics.

This book contains a wealth of material, critically arranged, and reads exceedingly well. It has an original approach and contains much novel material. The author specifically mentions those aspects of his book which he considers new. Among them are the following. (1) The hypothesis of H. Freudenthal (inaugural oration, Utrecht, 1946) according to which the Hindus, after having developed the decimal system with symbols for 1, ···, 9, 10, 20, ···, learned the sexigesimal system and the zero through Greek astronomy and thus acquired their own position system [compare O. Neugebauer, Quellen und Studien Ges. Math. Astr. Phys. Abt. B. Studien 3, 263–271 (1936), in particular, pp. 266–269]. (2) Arguments to show that Thales actually gave a logical proof of the four theorems ascribed to him by Eudemos. (3) A reconstruction of the Pythagorean number theory based on the arithmetical books of Euclid's Elements; the author claims that Book VII of the Elements should be ascribed to the Pythagoreans before Archytas [see the author, Math. Ann. 120, 127–153 (1948); these Rev. 9, 483]. (4) A comparison of the geometrical algebra of the "Elements" with Babylonian texts showing that the Greeks, in particular the Pythagoreans, have taken their material from the Babylonian tradition. (5) An attempt to show how Theodoros of Kyrene actually demonstrated the irrationality of what we call $\sqrt{2}, \sqrt{3}, \cdots, \sqrt{17}$, and why he stopped at $\sqrt{17}$. (6) A demonstration of certain deficiencies in Archytas' logical thinking by means of Book VIII of the "Elements." (7) An explanation of the mathematics in Plato's posthumous Epinomis, based on the work of Archytas. (8) An analysis of Book X of the "Elements," which is ascribed to Theaitetos. The reading of the book is facilitated by historical excursions and by many beautiful illustrations, including a map of the aqueduct of Samos and a reproduction of the Herculaneum mosaic of the death of Archimedes.

D. J. Struik (Cambridge, Mass.).

Chatterjee, Bina. Geometrical interpretation of the motion of the sun, moon and the five planets as found in the mathematical syntaxis of Ptolemy and in the Hindu astronomical works. J. Roy. Asiatic Soc. Bengal. Sci. 15, 41–89 (1949).

The author gives a summary of the main features of the Greek and Hindu descriptions of planetary motion. A comparison is made between the two theories and their similarities and differences are pointed out. Comparative tables of characteristic parameters are given.

O. Neugebauer (Providence, R. I.).

Morse, Marston. Trends in analysis. J. Franklin Inst. 251, 33–43 (1951).

Coolidge, J. L. The number *e*. Amer. Math. Monthly 57, 591–602 (1950).

*Voellmy, E. **Jost Bürgi und die Logarithmen.** Elemente der Math. Beiheft no. 5. Verlag Birkhäuser, Basel, 1948. 24 pp. (2 plates). 3.50 Swiss francs.

This is an account of Bürgi, the Swiss mathematician and mechanician who discovered logarithms and compiled a table of antilogarithms independently and almost contemporaneously with Napier. The work begins with a sketch of the history of the geometrical progression from Archimedes onwards, with allusions to Nemorarius, Saxo (died 1237 A.D.), Chuquet, Stifel (1487–1567), and others, including Vieta, Napier, Kepler, and Galileo. An interesting account of the life of Bürgi (1552–1632) follows, with his birthplace in Switzerland, his travels to Strassburg, Kassel, and Prague, and his friendship with Kepler, who admired him as a great mathematician, and with his brother-in-law and pupil Benjamin Bramer, to whom we owe many of the facts of his life and work.

Bürgi was an expert craftsman in clockwork and in astronomical devices prior to the invention of the telescope and pendulum clock. He was also a creative constructor of tables. The frontispiece and first page of his "Arithmetische und Geometrische Progress Tabulen . . ." [1620] are reproduced. [They are also printed in the Napier Tercentenary Memorial Volume, Longmans, Green, London, 1915, plate 12.] The title page of Bramer's "Bericht" [1648] which tells us of Bürgi is also reproduced. These tables of 1620 give entries of the numbers 0, 10, 20, 30, ··· up to 4000, on the first page and printed in red, against the numbers

100,000,000, 100,010,000, 100,020,001, ··· which are derived from a geometrical progression, and printed in black. The title page gives an epitome of the tables by arranging 47 red numbers 500, 1000, ···, 230,000, 230,270(022) around a circle against the corresponding black 105,126,407, 110,516,539, ···, 997,303,557, 1,000,000,000 around a concentric and smaller circle. Opposite the red 100,000 is the black 271,814,593.

The use of a precise decimal notation is discussed, as is also the question of the basis of the logarithmic scale. Bürgi definitely constructed the tables to facilitate multiplication and division, etc., and interpolated by proportional parts. The author considers that Bürgi invented the system in 1588, on the evidence of a remark of Reimarus Ursus Dithmarsus that Bürgi then possessed a means of lightening to an extraordinary degree the task of calculation. This antedates the earliest suggested discovery of logarithms by Napier, ~~1691~~ 1594. There is little reference to the considered reports in the Napier Memorial Volume [loc. cit.]. The picture of an unassuming genius is attractively drawn and the designation the "Swiss Archimedes" is not out of place.

H. W. Turnbull (St. Andrews).

Swinden, B. A. Geometry and Girard Desargues. Math. Gaz. **34**, 253–260 (1950).

Elenco delle pubblicazioni del prof. Gino Fano fino al luglio 1950. Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 33–45 (1950).

Terracini, Alessandro. Guido Fubini e la geometria proiettiva differenziale. Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 97–123 (1950).

Lecture given in September, 1948 at the 3d national congress of the Unione Matematica Italiana. Pages 115–123 contain a list of Fubini's publications.

Burckhardt, J. J. Obituary: Rudolf Fueter. Vierteljschr. Naturforsch. Ges. Zürich **95**, 284–287 (1950).

*Wheeler, Lynde Phelps. **Josiah Willard Gibbs. The History of a Great Mind.** Yale University Press, New Haven, Conn., 1951. xi+264 pp. (1 plate). $4.00.

Tosi, Armida. La "Nova stereometria doliorum vinariorum" di Keplero. Period. Mat. (4) **28**, 159–168 (1950).

*Hofmann, Jos. E. **Die Entwicklungsgeschichte der Leibnizschen Mathematik während des Aufenthaltes in Paris (1672–1676).** Leibniz Verlag, München, 1949. viii+253 pp.

*Hofmann, Joseph E. **Leibniz' Mathematische Studien in Paris. Leibniz zu seinem 300. Geburtstag, 1646–1946.** Lieferung 4. Walter de Gruyter & Co., Berlin, 1948. 70 pp. (4 plates).

Of these two works, which are intimately connected, the former is a sustained account of the mathematical discoveries of Leibniz, and particularly of what he accomplished during his four years in Paris. The central theme is the discovery of the calculus, the whole situation being very carefully studied, with a full historical account of what led up to it, covering the middle 17th century from the time of Fermat and Descartes, and an evaluation of what others achieved among both the forerunners and the contemporaries of Leibniz. The larger work, which gives the account and argument in considerable detail, contains the relevant mathematical results: the smaller is essentially an abridgement of the larger work, obtained by omitting most of the technical details. It is indeed a work appropriate to the occasion, that of a 300th anniversary in honour of Leibniz' birthday in 1646, and it is meant for a wider circle than that of mathematical readers. It is the fourth of a set of special contributions, being preceded by those which deal with other aspects of Leibniz' work [N. Hartmann, Leibniz als Metaphysiker, Leibniz zu seinem 300. Geburtstag, 1646–1946, Lieferung 1, de Gruyter, Berlin, 1946; E. Benz, Leibniz und Peter der Grosse, Lieferung 2, 1947; E. Hoschsetter, Zu Leibniz' Gedächtnis, Lieferung 3, 1948]; it is followed by a fifth contribution on Leibniz as logician [K. Dürr, Leibniz' Forschungen im Gebiet der Syllogistik, Lieferung 5, 1949]. This shorter work retains much of the biographical and psychological treatment that is to be found in the larger work. Both are very readable and illuminating, for the subject, that of the discovery of the calculus, is still something of a mystery. Among those who played a part in the drama of this discovery Barrow, Gregory, Newton, and Leibniz stand out preeminently, and the problem turns upon what it was that each discovered, and how far each was dependent upon another. The mystery is heightened by the rapidity with which two of these men of genius, Newton and Leibniz, reached to the heart of the matter; Newton in his early postgraduate years at Cambridge, and Leibniz ten years later at Paris: Newton starting under the tutelage of Barrow, and Leibniz under that of Huygens.

The author has told the story of this astonishing mathematical development of Leibniz with care and insight. He makes his hero live, and he draws largely for his material from contemporary documents and letters. His conclusion is this: that Leibniz had a better and sharper understanding of the fundamentals than had any of his contemporaries, and that he is to be held as the exclusive inventor of the calculus, and that in this creation he was influenced by no one else, except in the most trifling details. To the reviewer, this extreme view hardly follows from the facts that the author has so carefully stated, and the effect of the work would have been heightened had the author come to a more cautious conclusion.

Both works contain an excellent list of letters arranged chronologically, together with the sources where they may be consulted. Such letters range over the names of Descartes, Fermat, Wallis, Brouncker, Collins, Gregory, Barrow, Oldenburg, Leibniz, Huygens, Sluse, Newton, Tschirnhaus, Bernoulli, and others. Each statement, as it occurs in the text, is carefully referred to one or other letter: and in many cases the substance of a whole letter is given.

What distinguishes this investigation from the notable contributions in the past, by Cantor and others, is the place given to Gregory among the significant forerunners of Leibniz. The author accepts the evidence afforded by the recent (1932) discovery of manuscript notes of Gregory which shew that he had found, as he called it, 'a universal method' in the years 1668–1671 which included the general binomial theorem and also complicated instances to successively derived functions leading to Taylor's expansion of a function by means of a power series, and further the interpolation theorem. The author judges that this work of Gregory was essentially independent of that of Newton. If this is so it naturally affords evidence that Gregory also could be another independent discoverer of the infinitesimal

calculus. The possibility that Leibniz drew some inspiration from Barrow's Lectiones Geometricæ [London, 1670], a book which he possessed, is mentioned but dismissed, perhaps too lightly in view of the argument of J. M. Child to the contrary.

The early part of the story is especially interesting where the mathematical leaps from particularities to general principles are exhibited; especially beautiful is the passage from Pascal's arithmetical triangle to the harmonical triangle (where the entries are the reciprocals of the binomial coefficients) which is an early and highly original step taken by Leibniz and gives insight into his analytical power. Much is also told about the theory of equations and Leibniz' considerable work in algebra.

The mathematical world will be grateful for each of these books which are the fruits of prolonged scholarship; each is a mine of information and a guide to further study in an absorbingly interesting period in the history of mathematics.

H. W. Turnbull (St. Andrews).

Taton, René. Deux contributions de Monge à la création de la géométrie moderne. C. R. Acad. Sci. Paris **232**, 198–200 (1951).

Ein Kapitel aus der Vorlesung von Franz Neumann über mechanische Wärmetheorie. Königsberg 1854–55. Ausgearbeitet von Carl Neumann. Herausgegeben von Ernst Richard Neumann. Abh. Bayer. Akad. Wiss. Math.-Nat. Kl. (N.F.) no. **59**, 27 pp. (1950).

According to C. Neumann, the unpublished researches of F. Neumann, presented in his Königsberg lectures before 1850 (the year of the first memoir of Clausius on thermodynamics, but after the publications of Mayer (1842) and Joule (1843)), establish his priority as one of the founders of the mechanical theory of heat. F. Neumann's lectures, although worked over and prepared for the press by C. Neumann in 1895, have never been published; the present work, deriving from notes taken in 1854–1855 by H. Wild, is extracted from them, being the third lecture, in which the first law is explained.

Neumann begins by stating that the experiments presented in the preceding lectures show Carnot's view that the change Q in the quantity of heat is a function of caloric u alone is wrong; instead, as D. Bernoulli stated [Hydrodynamica, Dulseckeri, Argentorati, 1738, see Chapter X, §§ 1–6], temperature t is a manifestation of molecular motion. In § 1 Neumann considers the state of a substance as defined by a $t-u$ diagram; by first considering paths which are either isotherms or "caloric curves," then regarding a general path as approximated by a zig-zag sequence of such paths, he concludes that there is a function $f(t, u)$ such that $Q=\int_A^B f(t, u)\,du$, the integral being taken along the particular path connecting state A to state B. This he contrasts with the relation $Q=\Psi(u)$, which he attributes to Carnot. He does not define a caloric curve in this lecture; the quantity u is "the parameter of these curves." To the modern reader it is plain that the caloric curves are adiabates. In this connection it is interesting to notice the observation of LaMer [Science **109**, 598 (1949)], that if Carnot's two terms chaleur and calorique be rendered respectively as heat and entropy, rather than, as is customary, both confounded simply as heat, then the entire work of Carnot [Refléxions sur la puissance motrice du feu . . . , Bachelier, Paris, 1824] becomes consistent with the mechanical theory; that, contrary to the common statement found in textbooks, there is no error in Carnot's original treatment of the Carnot cycle, was remarked by Callendar [see the entry under Heat, Encyclopoedia Brittanica, 11th ed., 1910 to present eds., see §§ 16, 20, 22]. In § 2 Neumann states the First Law in general terms, hence in § 3 concluding the existence of a function $W(p, v)$, where v is the volume and p the pressure acting on a fluid body, such that (*) $dW=\mathfrak{A}f(t, u)\,du-p\,dv$, $\mathfrak{A}$ being the mechanical equivalent of heat. The function W, to be identified with the internal energy, Neumann calls Arbeitsvorrat. From (*) he concludes that $\mathfrak{A}(\partial f/\partial t)(\partial(u, t)/\partial(v, p))=1$. In § 4 Neumann derives from (*) the fact that it is possible to find a path, connecting two given states p_1, v_1 and p_2, v_2, along which no mechanical work is done, but that in general it is not possible to find a path such that all the energy change arises from mechanical work. Section 5 begins with an indication of how the Carnot cycle can be described in the present terms. By carrying two different substances through Carnot cycles with the same heat reservoirs at the same temperatures, Neumann concludes that it is plausible that if the quantity of heat absorbed by each substance when passing along the first isotherm is the same, then the mechanical work done by each in its full cycle is the same. This statement appears to be equivalent to the Second Law in the form enunciated earlier by Carnot [op. cit.]. Hence Neumann is able to conclude that $f(t, u)=g(t)h(u)$, where the function $g(t)$ is the same in form, up to a constant factor, for all substances. In § 6 he takes account of the kinetic energy of gross motion, stating at the end that when a body is brought to rest in a fluid, the temperature of the fluid must rise.

The reviewer hopes that the remainder of the lectures may be published. According to the introduction to the present one, the first contains an account of an experiment performed by F. Neumann in 1853, in which, by measuring the Joulean heat of a magnetically induced current, he obtained the value 429 kgm/cal. for the mechanical equivalent of heat.

C. Truesdell (Bloomington, Ind.).

Strong, E. W. Newton's "mathematical way." J. Hist. Ideas **12**, 90–110 (1951).

Una visione dell'opera scientifica di Tibor Radó. Rivista Mat. Univ. Parma **1**, 239–273 (1 plate) (1950).

***Severi, Francesco. Memorie Scelte. Vol. I.** Cesare Zuffi, Bologna, 1950. xx+458 pp. (1 plate).

The selected works of Severi, which will appear in four volumes, are being edited by B. Segre. Vol. 1 is divided into four sections containing papers on enumerative geometry, foundations of algebraic geometry, projective geometry, and the theory of modules and ideals.

Agostini, Amedeo. Il metodo delle tangenti fondato sopra la dottrina dei moti nelle opere di Torricelli. Period. Mat. (4) **28**, 141–158 (1950).

FOUNDATIONS

*Lorenzen, Paul. **Grundlagen der Mathematik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 11–22. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

A review of most of the work mentioned here has already appeared. In addition there should be mentioned the work of O. Becker on the interpretation of modal logic [Blätter für deutsche Philosophie **16**, 387–422 (1943)], that of Ackermann [Ein System der typenfreien Logik I, Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, N.F. no. 7, Verlag Hirzel, Leipzig, 1941], and the unpublished work of Lorenzen on the freedom of contradiction of the predicate calculus with a constructive domain of individuals; an account, too brief to be understandable, of the latter researches is given.

A. Heyting (Amsterdam).

Ackermann, Wilhelm. Widerspruchsfreier Aufbau der Logik. I. Typenfreies System ohne tertium non datur. J. Symbolic Logic **15**, 33–57 (1950).

The objective of the author is to obtain a consistent logistical system without any restriction of type. His previous attempt in this direction [Ein System der typenfreien Logik I, Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, N.F. no. 7, Verlag Hirzel, Leipzig, 1941] was based on a suggestion of Behmann [Jber. Deutsch. Math. Verein. **40**, 37–48 (1931)] that every predicate has a range of significance, and that the application of this predicate to an argument is a proposition only if the argument is in this range. In the present paper the author proposes a weakened propositional calculus which holds for any formulas formed by applying a predicate to an argument. In order to avoid paradoxes it is known [Curry, same J. **7**, 115–117 (1942); these Rev. **4**, 125] that in such a case implication must be weakened so as to omit either (a) the rule of inference (Abtrennungsregel), (b) the law $A \rightarrow A$, or (c) the law $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$. The author follows alternative (c). He motivates it by considering the syntactical nature of implication; accordingly $A \rightarrow (B \rightarrow A)$ and $A \rightarrow (B \rightarrow A \,\&\, B)$ are replaced respectively by $(\Gamma \rightarrow A) \rightarrow (B \rightarrow A)$ and $(\Gamma \rightarrow A) \rightarrow (B \rightarrow A \,\&\, B)$, where Γ is a particular asserted proposition. The postulates assumed are essentially those for a distributive lattice with this weakened implication giving the partial order; negation satisfies the law $A \,\&\, \bar{A} \rightarrow B$, the laws of double negation, and the de Morgan formulas, but the weakening of implication causes the transposition properties to fail; and quantification and abstraction $(\hat{x}(\cdots))$ satisfy what are essentially the usual properties. The postulate $(\Gamma \rightarrow A) \rightarrow A$ is admitted; but the converse holds only as a rule of inference. The deduction theorem is valid only in the modified form: if B is deduced from the premise A, then one of the formulas $A \rightarrow B$, $(\Gamma \rightarrow A) \rightarrow B$, $(\Gamma \rightarrow (\Gamma \rightarrow A)) \rightarrow B$, $\cdots$ holds.

This system the author proves to be consistent. The proof is reminiscent of Gentzen's Hauptsatz [Math. Z. **39**, 176–210 (1934)], in that the author shows that if certain derivable rules with composite conclusions are adjoined, then the Abtrennungsregel can be eliminated. The author discusses the Russell paradox and Cantor's theorem $2^{\mathfrak{n}} > \mathfrak{n}$. A certain arithmetic without excluded middle is derivable within the system. A paper is to follow giving certain generalizations and a reinstatement of the law of excluded middle with restrictions.

H. B. Curry.

Rosser, J. Barkley, and Wang, Hao. Non-standard models for formal logics. J. Symbolic Logic **15**, 113–129 (1950).

The first result in this paper is that the system Q of Quine [Amer. Math. Monthly **44**, 70–80 (1937)] has no standard model. More precisely, if Q is consistent, it has a model M in which $=$ of Q is represented by the equality relation in M while *Fin* (the sign in Q for the class of nonnegative finite cardinals $0, 1, \cdots$) is represented by the class of representatives of $0, 1, \cdots$. It follows that M is also a model for Q with the axiom of infinity added (Q'). In Q' a theory of ordinal numbers (*NO*) can be developed. It is shown that the class in M representing *NO* is not well ordered by the relation representing $\leqq$. The authors emphasize the fact that reasoning about the modelling of a logic L can be carried out within L, if L is one of the familiar formal logics. Actually they sketch a procedure for doing this, based upon methods of Tarski [Studia Philos. **1**, 261–405 (1935)] and Kemeny [same J. **13**, 16–30 (1948); these Rev. **9**, 487]. They prove now that if L is ω-consistent, the theorem "If L is consistent, then L has a standard model" cannot be proved within L, and neither within certain specified extensions of L.

A. Heyting (Amsterdam).

Wang, Hao. A theory of constructive types. Methodos **1**, 374–384 (1949).

Let $({}_*Z)$ be the system of number theory as formulated by Hilbert and Bernays [Grundlagen der Mathematik, vol. 1, Springer, Berlin, 1934, p. 465] with variables $x_{(1)}, y_{(1)}, \cdots$. Let $({}_*Z)'$ be a new system obtained from $({}_*Z)$ by adding a new kind of variables $x_{(2)}, y_{(2)}, \cdots$ which take as values classes of the elements of $({}_*Z)$ such that to every formula of $({}_*Z)$ there corresponds a class in $({}_*Z)'$. Identity for classes is defined by extensionality. Now repeat this process, extending $({}_*Z)'$ to a new system $({}_*Z)''$, introducing new variables $x_{(3)}, y_{(3)}, \cdots$ for classes of the elements and classes of $({}_*Z)'$, and so on. We thus obtain a system R with an infinite hierarchy of types in which the elements of the lowest type satisfy the axioms of number theory and the following axiom schema holds: "If n is any nonnegative integer and φ is a formula in which $y_{(n+1)}$ is not free and there is no free variable of type higher than $n+1$, no bound variable of type higher than n, then $(\exists y_{(n+1)})(x_{(n)})(x_{(n)} \varepsilon y_{(n+1)} \equiv \varphi)$ is an axiom." Then R is weaker than the theory of types, because no bound variable of type higher than n is allowed in φ, but is very similar to Weyl's system [Das Kontinuum . . . , Veit, Leipzig, 1918]. There is a sketch of a proof that if $({}_*Z)$ is consistent, then R has a denumerable model. The proof uses a method of Rosser and the author [see the preceding review].

I. L. Novak (Berkeley, Calif.).

Curry, Haskell B. A simplification of the theory of combinators. Synthèse **7**, 391–399 (1949).

This presents a simplified axiomatic treatment of the theory of combinators in combinatory logic. The combinators S and K of Schönfinkel are taken as primitive, rather than B, C, W, and either I or K, as in previous axiomatic treatments. The relation of the present system to previous axiom systems of the author and Rosser is described. The author discusses what is required of such a system in order that a completeness theorem may be provable. An alternative system is also given in which $BI = I$ occurs as an axiom

independent of the others, so that it may be replaced by weaker axioms if desired, as has been proposed by Rosser.

O. Frink (State College, Pa.).

Halldén, Sören. A reduction of the primitive symbols of the Lewis calculi. Portugaliae Math. **8**, 85–88 (1949).

It is shown that in the systems of strict implication of C. I. Lewis, the original three primitive operations, namely conjunction, negation, and possibility, may be defined in terms of a single ternary operation analogous to the stroke function of H. M. Sheffer. One such ternary operation, denoted by $[p, q, r]$, has the interpretation $-p \vee -q \vee -\Diamond r$. In the Lewis system S1 and in all the stronger systems this single operation will suffice. Similarly, the operation $\{p, q, r\}$, whose interpretation is $-p \cdot -q \cdot -\Diamond r$, may be taken as the single primitive operation in the system S4 and stronger systems, but not in the weaker systems S1, S2, S3.

O. Frink (State College, Pa.).

Martin, Norman M. Some analogues of the Sheffer stroke function in n-valued logic. Nederl. Akad. Wetensch., Proc. **53**, 1100–1107 = Indagationes Math. **12**, 393–400 (1950).

The author uses the term Sheffer function to denote a binary function in n-valued logic in terms of which all other functions may be defined. This is an important property of the Sheffer stroke function in the two-valued logic. Such a function may be taken as the single primitive function in an axiomatic treatment. In six theorems the author proves that nine different classes of functions which he defines are Sheffer functions in the n-valued logic for all n. These classes do not exhaust all the possibilities, and it is pointed out in particular that the author's theorems provide only eighteen Sheffer functions for the three-valued logic, whereas ninety such functions are known.

O. Frink (State College, Pa.).

Menne, Albert. Zur Wahrheitswertstruktur des Urteils. Methodos **1**, 390–404 (1949).

Diese Untersuchung betrifft die vier Urteilsformen A, E, I, O der aristotelischen Logik. Der Verfasser bewertet zunächst die Form "S ist P" mit drei Wahrheitswerten, nämlich W wenn SP=P, F wenn SP=0, und U sonst. In bezug auf die Verteilung dieser drei Werte über die vier Fälle "S ist P," "S' ist P," "S ist P'," und "S' ist P'" gibt es sieben mögliche Relationen zwischen S und P. Jeder dieser Relationen korrespondiert daher eine dreiwertige vierstellige Wertetafel. In bezug auf jede der vier Formen A, E, I, O, werden diese dreiwertigen Tafeln dadurch in zweiwertige umgeändert, dass man in den negativen Urteilen W und F vertauscht, dagegen in den positiven Urteilen sie unverändert lässt, weiterhin U in F für die universellen Urteile und in W für die partikulären Urteile umwandelt. Zu jeder der vier Formen gibt es dann mehrere Fälle, worin die erste Stelle ein W hat; also sind diese Formen nicht von einheitlicher Struktur. Der Verfasser untersucht die Regeln des logischen Quadrats und auch die Gesetze der Konversion und der Kontraposition; er zeigt dass alle klassisch richtigen Gesetze dieser Art aus seiner Formulierung herleitbar sind, allerdings unter der Voraussetzung, dass keine Klasse null oder die Allklasse ist. (Diese Deutung der aristotelischen Logik ist wohl nicht die einfachste; eine Deutung, worin alle klassisch richtigen Schlüsse ausnahmlos gelten, wurde von H. B. Smith [Symbolic logic, method and development, New York, 1927] gegeben.)

H. B. Curry (Brüssel).

Péter, Rózsa. Zum Begriff der rekursiven reellen Zahl. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 239–245 (1950).

Ausgehend von einem Aufsatz von Specker [J. Symbolic Logic **14**, 145–158 (1949); diese Rev. **11**, 151] nennt die Verfasserin eine Folge $a_1, a_2, \cdots$ von positiven rationalen Zahlen rekursiv, wenn $a_n = f(n)/g(n)$ gilt, wobei f und g rekursive Funktionen sind und $g(n) \geqq 1$ ist. Eine reelle Zahl r heisst rekursiv, falls es eine rekursive Folge a_n und eine rekursive Funktion $h(m)$ gibt derart, dass bei $n, n^* \geqq h(m)$ die Ungleichung $|a_n - a_{n^*}| \leqq 1/m$ gilt. Weiterhin sagt man, die reelle Zahl r definiere einen rekursiven Schnitt, wenn $m/(n+1) > r$ eine rekursive Beziehung zwischen m und n ist. Man kann dann leicht beweisen, dass r dann und nur dann einen rekursiven Schnitt definiert, wenn $[nr]$ (d.i. die grösste ganze Zahl die nicht grosser als nr ist) eine rekursive Funktion von n ist. Weiter kann man behaupten, dass jede einen rekursiven Schnitt definierende reelle Zahl r der folgenden Bedingung genügt: Ist b_n eine beliebige rekursive Funktion von n, so lässt sich r in eine Reihe $r = a_0 + a_1/b_1 + a_2/b_1b_2 + a_3/b_1b_2b_3 + \cdots$ derart entwickeln, dass a_n eine rekursive Funktion von n ist. Also ist jede solche Zahl r rekursiv. Specker hat gezeigt, dass die letzten Sätze sich nicht umkehren lassen. Die Hauptaufgabe dieses Aufsatzes ist, eine hinreichende Bedingung dafür anzugeben, dass eine rekursive Zahl r einen rekursiven Schnitt definiert. Diese Bedingung ist, dass r entweder rational oder irrational in dem Sinne ist, dass es rekursive Funktionen $f(p, q)$ und $g(p, q)$ gibt, so dass für alle positive ganze Zahlen p und q, und für $n \geqq g(p, q)$ gilt $|a_n - p/q| > 1/f(p, q)$.

H. B. Curry (Brüssel).

Moch, François. Ensembles, espèces et logique. C. R. Acad. Sci. Paris **232**, 201–202 (1951).

Malécot, G. La probabilité des causes et le problème de l'induction. Cahiers Rhodaniens **1**, 14 pp. (1949).

Starting with probability as a physically measurable quantity (frequency), the author shows that the problem of statistical estimation requires Bayes' theorem and a priori probabilities for its solution, which are in the last analysis "subjective." R. A. Fisher's method of optimum likelihood is not accommodated to the influence of subjective impressions of a priori probabilities, so that, in the author's opinion, it is valid only in its asymptotic form: in large samples. For a correct incorporation of such a priori impressions, one has to fall back on Bayes' formula, with its irreducible subjectivity. Whence come these a priori impressions? From our conviction that Nature follows simple laws, answers the author. But he points out that this conviction must not be regarded as based on a real property of Nature, but on human habits and limitations of thought, useful, but subjective.

B. O. Koopman (New York, N. Y.).

Martin, Norman M. The explicandum of the classical concept of probability. Philos. Sci. **18**, 70–84 (1951).

Popper, Karl R. Indeterminism in quantum physics and in classical physics. II. British J. Philos. Sci. **1**, 173–195 (1950).

ALGEBRA

***Vicente Gonçalves, J. Curso de álgebra superior. 2.° Vol. [Course of Higher Algebra. Vol. 2].** Lisbon, 1950. vi+512 pp.

This is a text on classical higher algebra. It contains chapters on polynomials and polynomial equations, matrices and determinants, invariant theory, the theory of symmetric functions, elimination theory, and the theory of algebraic functions. It is well written and is a rather interesting book, containing many exercises and a multitude of references to the literature of the subject. *A. A. Albert.*

***Perron, Oskar. Algebra. I. Die Grundlagen.** 3d ed. Walter de Gruyter & Co., Berlin, 1951. viii+301 pp. 16 DM.

***Perron, Oskar. Algebra. II. Theorie der algebraischen Gleichungen.** 3d ed. Walter de Gruyter & Co., Berlin, 1951. viii+261 pp. 14 DM.

New editions of texts which first appeared in 1932 and 1933, respectively. They are volumes 8 and 9 of Göschens Lehrbücherei, 1. Gruppe.

***Hasse, Helmut. Höhere Algebra. I. Lineare Gleichungen.** 3d ed. Sammlung Göschen Band 931. Walter de Gruyter & Co., Berlin, 1951. 152 pp. 2.40 DM.

***Hasse, Helmut. Höhere Algebra. II. Gleichungen höheren Grades.** 3d ed. Sammlung Göschen Band 932. Walter de Gruyter & Co., Berlin, 1951. 158 pp. 2.40 DM.

The 2d editions were in 1933 and 1937, respectively. Such changes have been introduced as were possible without altering substantially the arrangement of the pages.

Barnett, I. A., and Mendel, C. W. Generalized determinants of Vandermonde. Math. Z. **52**, 723–734 (1950).

Let $A=\|a_{ij}\|$ $(i, j=1, 2, \cdots, n)$ be a square matrix whose elements are independent real variables; let $D(A)$ denote the determinant of A; and let $a_{ij}^{(m)}$ denote the element in the ith row and jth column of the mth power of A. The determinant

$$\begin{vmatrix} a_{r_1 s_1}^{(m_1)} & a_{r_1 s_2}^{(m_1)} & \cdots & a_{r_1 s_n}^{(m_1)} \\ a_{r_2 s_1}^{(m_2)} & a_{r_2 s_2}^{(m_2)} & \cdots & a_{r_2 s_n}^{(m_2)} \\ \vdots & \vdots & & \vdots \\ a_{r_n s_1}^{(m_n)} & a_{r_n s_2}^{(m_n)} & \cdots & a_{r_n s_n}^{(m_n)} \end{vmatrix},$$

where $r_1, \cdots, r_n$ (the set r) and $s_1, \cdots, s_n$ (the set s) are arbitrary integers on the range 1 to n, equal or unequal, and where $0 \leqq m_1 < m_2 < \cdots < m_n$ are arbitrary integers, will be called a generalized determinant of Vandermonde and denoted by $V_s^r(A; m_1, m_2, \cdots, m_n)$. In part I the authors establish some simple factorizations for such generalized Vandermondes by means of the Hamilton-Cayley equation for the matrix A, and also some relations between the Vandermondes and certain Jacobians. Examples: (1) If $m \geqq 0$, the equation

$$V_s^r(A; m+1, m+2, \cdots, m+n) = D(A)\, V_s^r(A; m, m+1, \cdots, m+n-1)$$

holds identically in the elements of A. (2) The Jacobian of the coefficients of the characteristic equation with respect to the set of elements $a_{s_1 r_1}, a_{s_2 r_2}, \cdots, a_{s_n r_n}$ is, except possibly for sign, equal to $V_s^r(A; 0, 1, \cdots, n-1)$, where $r=(r_1, r_2, \cdots, r_n)$ and $s=(s_1, s_2, \cdots, s_n)$. Part II contains a proof of the following theorem: If A is a square matrix of order n, the Jacobian of the n traces of $A, A^2, \cdots, A^n$ (or the Jacobian of the coefficients of the characteristic function of A), with respect to each set of n distinct elements of A, at least one of which is a diagonal element, is never identically zero in the n^2 elements of A. The first proof of this theorem was given by the authors; the paper under review contains a shorter and more elegant proof due to O. Perron. Heineman's generalization of the classical Vandermonde [Trans. Amer. Math. Soc. **31**, 464–476 (1929)] is a special case of $V_s^r(A; m_1, m_2, \cdots, m_n)$. *G. B. Price.*

Shenton, L. R. A note on compound determinants. Math. Gaz. **34**, 133–135 (1950).

The author proves that determinants of order $n(n-1)$ of a special type can be written either as (a) a compound determinant of order $\frac{1}{2}n(n-1)$ whose elements are determinants of order 2, or as (b) a compound determinant of order $n-1$ whose elements are determinants of order n. The proofs use results of E. T. Whittaker on the representation of certain special determinants as compound determinants [Proc. Edinburgh Math. Soc. **36**, 107–115 (1918)].

G. B. Price (Lawrence, Kans.).

Collar, A. R. On the reciprocal of a segment of a generalized Hilbert matrix. Proc. Cambridge Philos. Soc. **47**, 11–17 (1951).

The inverse of the matrix $U(k)=[U_{pq}(k)]$, where

$$U_{pq}(k)=(k+p-q)^{-1}, \qquad p, q=1, 2, \cdots, n,$$

is shown to be $[V_{rs}(k)]$, where

$$V_{rs}=(-1)^{n+r-s-1}\frac{1}{k-r+s}\,\frac{(k-r+n)\cdots(k-r+1)}{(r-1)!(n-r)!}\times\frac{(k+s-1)\cdots(k+s-n)}{(s-1)!(n-s)!}$$

provided k is not an integer in the range $|k| \leqq n-1$. The problem of finding $U^{-1}(\frac{1}{2})$ arose in connection with the approximate representation of the aerodynamical load on swept-back aircraft wings. *G. B. Thomas.*

Boksenbom, Aaron S., and Hood, Richard. General algebraic method applied to control analysis of complex engine types. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **980**, 12 pp. (1950).

This is a discussion of the formulation of the fundamental equations for a linear automatic control system with several degrees of freedom. The particular systems which are of chief interest to the authors are complex gas turbine engines. The theory is presented in terms of matrix algebra, without any essential novelties. Considerable attention is given to the case in which the system is so constructed that the several variables are controlled independently.

L. A. MacColl (New York, N. Y.).

Sueoka, Seiichi. On eigenvalues of incompletely reducible matrix. J. Phys. Soc. Japan **4**, 361–363 (1949).

Let H_p be a square Hamiltonian matrix of order p. Let

$$H_{n+m}=\begin{pmatrix} H_n & A \\ \bar{A} & H_m \end{pmatrix},$$

where $\bar{A}$ is the transposed conjugate of A. Let U_q be a unitary matrix such that $U_q^{-1}H_qU_q=\Lambda_q$ is in diagonal form and let E_q be the identity matrix of order q $(q=n, m)$. If λ

is an eigenvalue of H_{n+m} but not of H_n, then

$$\det\{\lambda E_m - \Lambda_m - \bar{B}(\lambda E_n - \Lambda_n)^{-1}B\} = 0,$$

where $B = U_n^{-1}AU_m$. The case $m=1$ is attributed to L. Brillouin [J. Phys. Radium (7) 3, 373–389 (1932), p. 379]. *G. B. Thomas* (Cambridge, Mass.).

Vicente Gonçalves, J. Calcul abrégé d'un résultant. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 1, 201–204 (1950).

A method is given which shortens the computation of the resultant of two polynomials. *E. Frank* (Chicago, Ill.).

Laureana, Giuseppe. Eliminante nodale. Boll. Un. Mat. Ital. (3) 5, 133–139 (1950).

Suppose $m+1$ forms are given, homogeneous in several systems of coordinate variables and in r sets of parameters, and that the degrees in the parameters are the same for all forms of the set. The author determines the characters of the nodal eliminant, which is obtained by eliminating the parameters. *J. A. Todd* (Cambridge, England).

Abstract Algebra

Kimura, Naoki. Independency of axioms of lattices. Kōdai Math. Sem. Rep. 1950, 14 (1950).

It is known that a lattice can be defined as a system with two binary, commutative, and associative operations, related by two contractions laws. The author shows that these six identities are independent. *G. Birkhoff.*

Lombardo-Radice, Lucio. Sull'involuzione equatoriale del reticolo distributivo libero con n generatori. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 212–219 (1950).

The author considers the "natural" involution (dual automorphism of period two) $\delta: x \to \delta(x)$ of the free distributive lattice L_n generated by n symbols $a_1, \cdots, a_n$, through the correspondence $\delta(a_i) = a_i$. All "self-dual" elements, with $\delta(x) = x$, have dimension $\frac{1}{2}d[L]$. The author interprets the condition $\delta(x) = x$ in terms of the known representations $L_n = 2^{2^n}$, and computes all twelve self-dual elements in the case $n=4$. *G. Birkhoff* (Cambridge, Mass.).

Iseki, Kiyoshi. The theorems on one-to-one mapping of lattices by C. E. Rickart. J. Osaka Inst. Sci. Tech. Part I. 1, 127–128 (1949).

The author proves that any one-to-one meet preserving map of one lattice onto a second is also join preserving and observes that this result contains two theorems proved by the reviewer [Bull. Amer. Math. Soc. 54, 758–764 (1948); these Rev. 10, 96]. This same observation has been made in correspondence with the reviewer by a number of mathematicians and has, in fact, already been published by Dilworth and Ward [ibid. 55, 1141 (1949); these Rev. 11, 309]. *C. E. Rickart* (New Haven, Conn.).

Michiura, Tadashi. On characteristic properties of Boolean algebras. J. Osaka Inst. Sci. Tech. Part I. 1, 129–133 (1949).

It is noted that each of the following conditions is sufficient for a distributive lattice L to be a Boolean algebra: (i) Every principal ideal is simple; (ii) every principal ideal is the last-residue-class of its last-residue-class [cf. V. Krishnan, Proc. Indian Acad. Sci., Sect. A. 16, 176–190 (1942); these Rev. 4, 129]; (iii) every principal ideal is the intersection of its maximal ideal divisors. The necessity of these conditions, for L to be a Boolean algebra, was already known. *G. Birkhoff* (Cambridge, Mass.).

Snapper, E. Completely primary rings. II. Algebraic and transcendental extensions. Ann. of Math. (2) 53, 125–142 (1951).

This is a continuation of a previous paper by the same author [same Ann. (2) 52, 666–693 (1950); these Rev. 12, 314], and uses the same terminology and notation. In particular, R and S will always denote completely primary rings. If $N = N(R)$ is the radical of R, then the field R/N is denoted by $\bar{R}$. If $R \subset S$, and B is any subset of S, $R(B)$ denotes the smallest completely primary subring of S which contains R and B. If $\sigma \in S$, $R(\sigma)$ is a simple extension of R and is algebraic with respect to R if σ satisfies at least one regular polynomial of $R[x]$. Otherwise, that is, if every polynomial of $R[x]$ satisfied by σ is nilpotent, $R(\sigma)$ is a simple transcendental extension of R. If $R \subset S$, then S can be obtained from R by a (possibly infinite) number of simple algebraic and transcendental extensions; in fact, S can be obtained by a purely transcendental extension followed by an algebraic extension. The ring S is a principal extension of R if $N(S)$ is the smallest ideal in S which contains $N(R)$. This concept plays an important role in the theory. If S is a simple transcendental extension of R, then S is a principal extension of R, while simple algebraic extensions are not necessarily principal. It turns out that extensions of R are intimately related to corresponding field extensions of $\bar{R}$. For example, $R(\sigma)$ is respectively an algebraic or transcendental extension of R if and only if $\bar{R}(\bar{\sigma})$ is an algebraic or transcendental extension of $\bar{R}$. The author extends to completely primary rings most of the familiar theory of algebraic and transcendental extensions of fields. A typical result is the following generalization of Lurōth's theorem. If S is a simple transcendental extension of R such that $N(S) = N(R)$ and $R \subset R_1 \subset S$, then either $R = R_1$ or R_1 is a simple transcendental extension of R with $N(R_1) = N(R)$. The same result holds without restrictions on the radicals, provided $N(R)$ is nilpotent. *N. H. McCoy* (Northampton, Mass.).

Jacobson, N., and Rickart, C. E. Jordan homomorphisms of rings. Trans. Amer. Math. Soc. 69, 479–502 (1950).

Étant donnés deux anneaux associatifs A, B, un "homomorphisme de Jordan" $a \to a^J$ de A dans B est une fonction additive telle qu'en outre $(a^2)^J = (a^J)^2$ et $(aba)^J = a^J b^J a^J$. Continuant l'étude de ces applications, commencée par Ancochea [Ann. of Math. (2) 48, 147–153 (1947); ces Rev. 8, 310], Kaplansky [Duke Math. J. 14, 521–525 (1947); ces Rev. 9, 172], et Hua [Proc. Nat. Acad. Sci. U. S. A. 35, 386–389 (1949); ces Rev. 10, 675], les auteurs cherchent des conditions moyennant lesquelles un homomorphisme de Jordan est un homomorphisme ordinaire, ou un antihomomorphisme, ou une "somme directe" $a \to a^{J_1} + a^{J_2}$ d'un homomorphisme de A dans B_1 et d'un antihomomorphisme de A dans B_2 lorsque B est somme directe de deux anneaux B_1, B_2. Ils montrent que l'on est dans l'un des deux premiers cas si B n'a pas de diviseur de 0, ou lorsque B est un anneau primitif (au sens de Jacobson) ayant des idéaux minimaux et que $a \to a^J$ applique A sur B; on est dans le troisième lorsque A est un anneau simple (distinct d'un corps) ayant des idéaux minimaux, ou plus généralement lorsque A est un anneau "localement matriciel," c'est-à-dire tel que toute partie finie de A puisse être plongée dans un

sous-anneau de A isomorphe à un anneau de matrices finies sur un anneau convenable. Parmi les méthodes utilisées, mentionnons en particulier celles qui consistent à introduire l'anneau de Jordan (resp. l'anneau de Lie) obtenu en remplaçant dans A la multiplication par $\{ab\}=ab+ba$ (resp. $[ab]=ab-ba$); on a en effet $[[ab]c]'=[[a'b']c']$ et, si la division par 2 est possible, $\{ab\}'=\{a'b'\}$. Les auteurs sont ainsi amenés à étudier ces structures (et en particulier les idéaux correspondants) dans le cas où A est un anneau "localement matriciel" ou un anneau primitif ayant des idéaux minimaux. *J. Dieudonné* (Baltimore, Md.).

Hall, Marshall, Jr. A basis for free Lie rings and higher commutators in free groups. Proc. Amer. Math. Soc. 1, 575–581 (1950).

Let L be the free Lie-ring with q generators $x_1, x_2, \cdots, x_q$ and with $[x, y]$ as the symbol for the product. The number of linearly independent elements of degree n in L has been determined by E. Witt [J. Reine Angew. Math. 177, 152–160 (1937)]. Let F be the free group on q generators and F_n its subgroup generated by the complex commutators of weight n. P. Hall [Proc. London Math. Soc. (2) 36, 29–95 (1933); especially pp. 67 ff.] has described a systematic "collecting process" for complex commutators of increasing weight. Since there exists a well-known isomorphism between the additive group of the homogeneous elements of degree n in L and the multiplicative group of F_n/F_{n+1} the problem of finding a linearly independent basis is the same in both cases. The author gives an explicit construction of such a basis and shows that in the group case exactly those commutators which occur in Hall's collecting process form a basis. Monomials in $x_1, x_2, \cdots, x_q$ are defined to be "standard" by recurrence: (i) The standard monomials of degree 1 are $x_1, x_2, \cdots, x_q$. Let the standard monomials of degree $\leqq n-1$ be defined and arranged in simple order so that, if degree $u<$degree v, then $u<v$. (ii) $[u, v]$ of degree n with degree $u=r$, degree $v=s$, $r+s=n$ is standard, if both u and v are standard and $u>v$; and if $u=[x, y]$ in standard form (i.e., as linear combination of standard monomials), then $v\geqq y$. The main result is then the theorem: The standard monomials form a linearly independent basis for the free Lie ring L. *K. A. Hirsch* (Newcastle-upon-Tyne).

Noether, Emmy. Idealdifferentiation und Differente. J. Reine Angew. Math. 188, 1–21 (1950).

The definition given by the author of the different of a ring with respect to a subring by means of a process of "idealdifferentiation" was first made public in a lecture held at the meeting of the Deutsche Mathematische Vereinigung (Prague, 1929). The author died before she completed writing the paper which she probably intended to publish on the subject; the editors of the J. Reine Angew. Math. decided to print the partial manuscript of this intended paper which was put at their disposal by H. Grell. The first 6 sections of the paper are almost entirely complete; sections 6 and 7 contain only an outline of the results and proofs, but this outline is precise enough to make the projected content perfectly clear.

The paper begins with a definition of what is now called the tensor product of two algebras A and B over a ring C; it is assumed that C has a unit element, which is also unit element of A and B. The definition which is given differs from the usual one only by the requirement that A and B should have exactly C as intersection in their product. Under this condition, the product does not always exist, but certainly does if one of the rings A and B has a (linearly independent) base over C.

Let A be a commutative algebra over a ring C which has a unit element in C. Assume that we can form the product $A\times A'$ of A with an algebra A' over C which is isomorphic to A itself; let f be an isomorphism of A with A'. By the difference ideal is meant the ideal $\mathfrak{P}$ of $A\times A'$ generated by the elements $x-f(x)$, $x\in A$; by the difference quotient is meant the ideal $\mathfrak{A}=\{0\}:\mathfrak{P}$ of $A\times A'$, composed of the elements a such that $a\mathfrak{P}=\{0\}$. The different of A with respect to C is then defined to be the image of $\mathfrak{A}$ under the mapping of $A\times A'$ onto A defined by $x\to x$, $f(x)\to x$ $(x\in A)$.

The case where A is given (over C) by generators and relations is considered, and it is shown how to obtain the different by means of the ideal of relations. In particular, if $A=C[z]$, z integral over C and root of an equation $G(Z)=0$ (G being a polynomial with coefficients in C and leading coefficient 1), then the different exists (i.e., $A\times A'$ exists) and is the ideal $(G'(z))$. In case A can be split into the direct sum of algebras over C which have differents, it is shown that A has a different which is the direct sum of the differents of the components.

Let L be a finite algebraic separable extension of a field K, and let L' be an isomorphic copy of L. The notion of the different of L with respect to K is of course trivial; but not that of the difference ideal or the difference quotient. It is proved that $L\times L'$ is the direct sum of the difference ideal and the difference quotient. Let also M be the Galois extension of K determined by L. If L_i $(1\leqq i\leqq m)$ are the conjugates of L in M, then each $L\times L_i$ may be considered as a subring of $L\times M$. It is proved that $L\times M$ is the direct sum of the extensions to it of the difference quotients of the rings $L\times L_i$, and that the intersection of the extensions of the difference ideals of the rings $L\times L_i$ is $\{0\}$. The extensions $\mathfrak{A}_i$ of the difference quotients are the ideals generated by the primitive idempotents e_i of $L\times M$. The notion of complementary bases is then defined in terms of these idempotents: if $\{t_1, \cdots, t_n\}$ is a base of L, the complementary base in L_i is formed by the coefficients of the expression of e_i in terms of $t_1, \cdots, t_n$ with coefficients in L_i. The equivalence of this definition with the classical one in terms of traces is established.

The notation being as above, assume now that C is an integrally closed subring of K and that A is an "order" of L with respect to C. Then it is shown that the different exists and is an ideal $\neq\{0\}$. Let A' be an isomorphic copy of A and L' its field of quotients; then the product $A\times A'$ is a subring of $L\times L'$. On the other hand, $L\times L'$ is contained in $L\times M$, and there corresponds to L' a primitive idempotent e of $L\times M$. The different of A with respect to C is then proved to be the set of all $x\in L$ such that $xe\in A\times A'$. It is shown that this definition of the different coincides with the one in terms of complementary modules, and also with the definition in Kroneckerian style (when it applies) in terms of a defining equation $G(U)=0$ for a "generic element" u of A (containing indeterminates): the different is then the ideal content of the polynomial $G'(u)$ in the coefficients of u.

In case C is the ring of rational integers (or the quotient ring of a prime, if one has relative fields in view) and A is the maximal order of a field of algebraic numbers, then, if p is a prime of C, and t an exponent large enough, the different of A goes (mod p^t) into the different of A/p^tA over C/p^tC. This result allows one to give explicit formulas for the computation of the different, because A/p^tA may be obtained from C/p^tC by adjunction of a single element.

The paper does not contain the proof of the multiplication formula for relative differents, although the author obviously intended to include such a proof, as evidenced by the introduction.

To conclude, the reviewer wants to point out the obvious analogy between this definition of the different and the introduction of canonical classes in algebraic geometry as defined by self-intersections on product varieties. Certain indications given in the introduction may hint at the possibility that the author intended to push her investigations further in this direction. Whether this was the case or not, the present paper, although published fifteen years after the death of the author, proves once more how much a forerunner she was in all branches of algebra.

C. Chevalley (New York, N. Y.).

Raffin, R. Algèbres non associatives. Algèbres du quatrième degré. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 574–578 (1950).

If a nonassociative algebra A of fourth degree (over an associative, commutative ring B with unity) has unique cubes and satisfies a certain "condition (L)," then A has commutative powers. *R. D. Schafer* (Philadelphia, Pa.).

Zemmer, Joseph L., Jr. A note on division algebras of order sixteen. Portugaliae Math. **9**, 171–176 (1950).

Another negative result in the search for real (nonassociative) division algebras of order 16. It is known that the existence of a "norm function" for an algebra would insure that it is a division algebra. The author shows that, over the rational numbers and the real numbers, none of the algebras in a class discussed by the reviewer [Bull. Amer. Math. Soc. **52**, 874–881 (1946); these Rev. **8**, 193] possesses a norm function. *R. D. Schafer* (Philadelphia, Pa.).

Theory of Groups

Schwarz, Štefan. On various generalizations of the notion of a group. Časopis Pěst. Mat. Fys. **74** (1949), 95–113 (1950). (Czech. English summary)

Expository paper.

Thierrin, Gabriel. Sur une condition nécessaire et suffisante pour qu'un semi-groupe soit un groupe. C. R. Acad. Sci. Paris **232**, 376–378 (1951).

By definition, a semi-group possesses an associative binary operation and satisfies both cancellation laws. The following theorem is proved: A semi-group S is a group if and only if S satisfies the condition: (α) For each x in S, there is an element x' in S such that $xx'x=x$. Examples are given to show that the condition (α) implies neither the associative law nor the cancellation laws. *F. Kiokemeister.*

Michiura, Tadashi. On a definition of lattice-ordered groups. II. J. Osaka Inst. Sci. Tech. Part I. **1**, 117–119 (1949).

[For part I see the same vol., 27 (1949); these Rev. **11**, 497.] The author gives various postulate systems for lattice-ordered groups, which include group postulates and sets of conditions on the unary operation $a\to a^*$ (a^* signifies $a\cup 0$). His most interesting set of conditions is perhaps (1) $(a^*)^*=a^*$, (2) $a-b=-(b^*-a)^*+(a^*-b)^*$, and (3) $(a+x-a)^*=a+x^*-a$. Condition (1) can be replaced by $(-(a^*))^*=0$. *G. Birkhoff* (Cambridge, Mass.).

***Haefeli-Huber, Verena Esther. Ein Dualismus als Klassifikationsprinzip in der abstrakten Gruppentheorie.** Thesis, University of Zürich, 1948. 132 pp.

This thesis carries out a generalization of two papers [P. Hall, J. Reine Angew. Math. **182**, 130–141, 206–214 (1940); these Rev. **2**, 211, 125] as suggested by P. Hall. Let $f=f(x_1, \cdots, x_n)$ be an arbitrary word on $x_1, \cdots, x_n$ and their inverses. In a group G, the word group $W_f(G)$ is the subgroup generated by all values of f when the x's vary over G. The period group $M_f(G)$ consists of all elements ξ such that

$$f(\xi x_1, x_2, \cdots, x_n)=f(x_1, \xi x_2, \cdots, x_n)=\cdots$$
$$=f(x_1, x_2, \cdots, \xi x_n)=f(x_1, x_2, \cdots, x_n)$$

for all $x_1, \cdots, x_n$. Both the word group and the period group are characteristic, and the word group is even fully invariant, though the period group need not be. The classification given in this thesis is the generalization of isoclinism as given by P. Hall for the case in which f is the commutator $x_1^{-1}x_2^{-1}x_1x_2$; the word group will then be the commutator subgroup and the period group will be the center. We say that two groups G and G' are in the same isoclinic family if (1) $G/M_f(G)$ and $G'/M_f(G')$ are isomorphic, (2) $W_f(G)$ and $W_f(G')$ are isomorphic, and (3) these two isomorphisms are such that if cosets correspond in (1) the words f formed from coset representatives will correspond in (2). Groups are classified according to the isoclinic family to which they belong. The treatment is very general and contains several hundred definitions and no examples. Many relationships are found depending on these and derived concepts.

M. Hall (Washington, D. C.).

Černikov, S. N. On the condition of minimality for Abelian subgroups. Doklady Akad. Nauk SSSR (N.S.) **75**, 345–347 (1950). (Russian)

The author [same Doklady (N.S.) **63**, 11–14 (1948); these Rev. **10**, 590] has shown that if P is a locally finite p-group which satisfies the descending chain condition for Abelian subgroups, then P is a special group, i.e., an extension with finite factor-group of the direct product of a finite number of groups of type p^∞. The following theorems concern similar ideas. (1) Let P be a locally finite p-group, with no subgroup of type p^∞. If some maximal Abelian subgroup of P is finite, then P is itself finite. (2) Let P be a p-group with ascending central series. If some maximal Abelian subgroup of P is special, then P is special. (3) If P is a p-group, having a maximal Abelian subgroup which is special, and if P satisfies (*), then P has an ascending central series and is therefore special. Here (*) is called "the normalisor condition"; this seems to be the condition that every Abelian subgroup is properly contained in its normalisor. *J. L. Brenner.*

Amato, Vincenzo. Curve algebriche a gruppo di monodromia G_n. Matematiche, Catania **5**, 91–97 (1950).

If $m=\rho r$, and if S is the permutation,

$$(u_1 \cdots u_r)(u_{r+1} \cdots u_{2r}) \cdots (u_{(\rho-1)r+1} \cdots u_m),$$

on the m letters $u_1, \cdots, u_m$, let G_S denote the normalizer of S in the symmetric group $G_{m!}$. Then G_S is of order $r^\rho \cdot \rho!$ and has a normal Abelian subgroup Γ_S of order r^ρ with factor group G_S/Γ_S isomorphic to the symmetric group $G_{\rho!}$. Equations of order m with Galois group G_S are constructed

with the aid of resolvents of order ρ and Galois group $G_{\rho!}$. By the same means the ramification properties of an algebraic function $u(z)$ with monodromy group G_8 are discussed.
H. T. Muhly (Iowa City, Iowa).

Neumann, B. H. A two-generator group isomorphic to a proper factor group. J. London Math. Soc. **25**, 247–248 (1950).

Let F be the free group with two generators, and if H is a subgroup of F, denote by H^F the least normal subgroup of F containing H. It has been shown [G. Higman, the author, and H. Neumann, same J. **24**, 247–254 (1949); these Rev. **11**, 322] that F has a subgroup E with an infinite set $e_1, e_2, \cdots$ of free generators such that if R is a normal subgroup of E, then $R^F \cap E = R$. In particular, if R is a proper normal subgroup of E, then R^F is a proper subgroup of E^F. In this paper the author shows that there is an automorphism of F mapping e_i on e_{i+1}, $i=1, 2, \cdots$, and therefore mapping E^F on R^F, where R is the normal subgroup of E normally generated by $e_2, e_3, \cdots$. It follows that F/E^F is isomorphic to F/R^F, though R^F is a proper subgroup of E^F. That is, F/R^F is a two-generator group which is isomorphic to its proper factor group $(F/R^F)/(F/E^F)$.
G. Higman (Manchester).

Higman, Graham. A finitely related group with an isomorphic proper factor group. J. London Math. Soc. **26**, 59–61 (1951).

The surprising fact that a finitely generated group which is defined by finitely many relations may be isomorphic to a proper quotient group is established by producing the following simple example. The group G is generated by elements a, b, c subject to the relations $a^{-1}ca = b^{-1}cb = c^2$. Adding to these relations the relation $aca^{-1} = bcb^{-1}$ one obtains a proper quotient group isomorphic to G.
R. Baer (Urbana, Ill.).

Higman, Graham. A finitely generated infinite simple group. J. London Math. Soc. **26**, 61–64 (1951).

Suppose that the group G is generated by elements a, b, c, d subject to the relations: $a^{-1}ba = b^2$, $b^{-1}cb = c^2$, $c^{-1}dc = d^2$, $d^{-1}ad = a^2$. This group G is shown to be infinite and to be without normal subgroups of finite index, except G. As a finitely generated group G possesses maximal normal subgroups $N \neq G$; and it is clear that G/N is an infinite simple group.
R. Baer (Urbana, Ill.).

Gol'fand, Yu. A. On the group of automorphisms of the holomorph of a group. Mat. Sbornik N.S. **27(69)**, 333–350 (1950). (Russian)

Let G be a group, A its group of automorphisms, Γ its holomorph, Ω the group of automorphisms of Γ, Δ the subgroup of Ω leaving G invariant, Σ the group of inner automorphisms of Γ. A mapping $u(\alpha)$ of A into G is a crossed character if $u(\alpha\beta) = u^\beta(\alpha)u(\beta)$; u is regular if $\alpha \to \alpha\varphi[u(\alpha)]$ is an automorphism of A, where $\varphi(x)$ denotes the inner automorphism by x. For v in G, the function $u(\alpha) = v^{-\alpha+1}$ is a principal crossed character. The regular crossed characters form a group U, and the principal ones a normal subgroup V of U. The groups Δ and Σ are semi-direct products of A with U and V, respectively. In special cases more can be said. If G is noncommutative and characteristically simple, then $\Omega:\Gamma = 2$. If G is cyclic of odd order, or Abelian of type $(p, p, \cdots, p)$ with p odd, then Γ is complete, that is, it is a centreless group with all its automorphisms inner.
I. Kaplansky (Chicago, Ill.).

Siegel, Carl Ludwig. Bemerkung zu einem Satze von Jakob Nielsen. Mat. Tidsskr. B. **1950**, 66–70 (1950).

Using geometric methods Nielsen [Mitt. Math. Ges. Hamburg **8**, part 2, 82–104 (1940); these Rev. **2**, 213] proved the following theorem. Let F be a non-Abelian group of linear fractional transformations of the unit circle $|w| < 1$ onto itself, such that every transformation except the identity is hyperbolic. Then F is discontinuous. The author generalizes this result, using an elementary algebraic proof, to the following, where F operates on the upper half plane: Assume that F does not take into itself a set consisting of one or two real points and that F does not contain infinitesimal elliptic elements. Then G is discontinuous in the interior of the upper half plane.
D. Montgomery.

Gotô, Morikuni. On the group of formal analytic transformations. Kôdai Math. Sem. Rep. **1950**, 45–46 (1950).

The reviewer and W. T. Martin [Several Complex Variables, Princeton University Press, 1948; these Rev. **10**, 366] introduced the group E_k of formal transformations $x_j' = \sum a_j^l x_l +$ (higher powers) in terms of (nonconvergent) power series with complex coefficients and with a suitable weak topology. The author obtains the following results: A locally compact subgroup of E_k is a Lie group and the subgroup $x_j' = x_j +$ (higher powers) is (topologically) solvable.
S. Bochner (Princeton, N. J.).

Vrănceanu, G. Classification des groupes de Lie de rang zéro. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1**, 40–86 (1950). (Romanian. Russian and French summaries)

A method is indicated to put in a partially canonical form the equations of structure of a nilpotent Lie algebra. In the case of algebras of dimension 6, the reduced formulas indicated by the author still contain 2 arbitrary constants, while there are only a finite number of types of nilpotent algebras of dimension 6.
C. Chevalley (New York, N. Y.).

Calabi, Lorenzo. Les groupes de Lie réels à quatre dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 579–586 (1950).

This paper contains a classification of all simply connected Lie groups of dimension 4 (for groups which contain a simple subgroup, not only the simply connected forms are given, but the groups are completely classified). It should be observed that, in spite of the promise made to give the center in each case, the classification does not mention the discrete centers of some of the groups which are given (for instance the group G_3 of dimension 3 and the group (13) of dimension 4). On the other hand, the usefulness of the results would have been improved if the author had stated explicitly the relationship between the coordinate system in terms of which the composition formulae are given and the exponential mapping in the group.
C. Chevalley.

Nomizu, Katsumi, and Gotô, Morikuni. On the group of automorphisms of a topological group. Tôhoku Math. J. (2) **2**, 47–50 (1950).

Soit G un groupe localement compact, $A(G)$ le groupe des automorphismes de G muni de la topologie introduite

par G. Birkhoff (topologie induite par la topologie de la convergence uniforme dans l'espace compact obtenu en complétant G par un "point à l'infini"). Ils démontrent le théorème suivant: Si G est un groupe de Lie connexe, pour que $A(G)$ soit compact, il faut et il suffit que G soit compact et ait un centre de dimension $\leqq 1$. Ils montrent par ailleurs qu'il existe des groupes abéliens compacts G de dimension finie (resp. infinie) pour lesquels $A(G)$ est fini (resp. compact).

J. Dieudonné (Baltimore, Md.).

Kuranishi, Masatake. On conditions of differentiability of locally compact groups. Nagoya Math. J. **1**, 71–81 (1950).

Let G be a locally compact group and U a neighborhood of the identity. If for x in U, $x^2, x^3, x^4, \cdots, x^n$ lie in U while x^{n+1} is not in U, let $\delta_U(x)=n$ and if no such n exists let $\delta_U(x)=\infty$. The group G is said to have property S if there exists a U such that $\delta_U(x)$ is finite if x is in $U-e$. If G has property S there is defined on a subset Φ^U of U the Lie function sequence $F_n(x, y)=(x^{r-n}y^{r-n})^{r^n}$. Theorem 1 says that G is a Lie group provided the function sequence has a subsequence converging to an $F(x, y)$ which determines a local Abelian group. Theorem 2 states that G is a Lie group if there is a uniformly convergent subsequence. Theorem 3: Let G be a locally compact group in which there is a metric d for a neighborhood V satisfying the following condition with positive constants K_1, K_2, K_3, K_4: (E) If x, y, xy are in V, then $K_2d(y, e)\leqq d(xy, x)\leqq K_1d(y, e)$; (F) if n is less than or equal to $\delta_V(x)$, $\delta_V(y)$, then $K_4 2^n d(x, y)\leqq d(x^{2^n}, y^{2^n})\leqq K_3 2^n d(x, y)$. Then G is a Lie group. Theorem 4. Let G be a strongly effective locally compact group of transformations of class C on a manifold of class C. Then G is a Lie group.

D. Montgomery.

Gleason, Andrew M. Arcs in locally compact groups. Proc. Nat. Acad. Sci. U. S. A. **36**, 663–667 (1950).

The author gives an outline of the proof of the following theorem: Every locally compact group which is not totally disconnected contains an arc. To prove this, the author first defines a locally compact uniform structure on the family C of all compact subsets of a locally compact uniform space G. This structure is a generalization of the Hausdorff metric of the closed subsets of a metric space. If G is in particular a locally compact group, and if the product of two compact subsets of G is defined as usual, C becomes a topological semigroup with respect to that topology. Let S be the subsemigroup of those compact sets which contain the identity e. Then S is closed in C and is again a locally compact semigroup. Moreover, by the inclusion relation $A\supset B$, an order is defined in S, so that it is an ordered semigroup.

Now suppose that G is metrizable, connected, and contains more than one element. Take a fundamental system $\{X_n\}$ of compact neighborhoods of e in G; then $X_n\to e$ in S. Since G is connected, each X_n generates a subsemigroup which is not bounded in S. As in the case of locally compact groups [D. Montgomery, Ann. of Math. (2) **49**, 110–117 (1948); these Rev. **9**, 496], the author proves the following lemma: A group G of the above-mentioned structure contains either arbitrarily small connected, compact subgroups different from e or a family $F(t)$ $(0\leqq t\leqq 1)$ of connected, compact subsets such that (i) $F(t)\neq e$ if $t>0$, (ii) $\bigcap_{t>0}F(t)=F(0)=e$, and (iii) $F(t)F(u)=F(t+u)$ if $t+u\leqq 1$. The theorem mentioned at the beginning follows then easily from this lemma. Refining some results of Montgomery [ibid. **50**, 570–580 (1949); **48**, 650–658 (1947); these Rev. **11**, 10; **9**, 174], and applying the main theorem, the author also proves that every arcwise connected subgroup of a Lie group is analytic and that any connected locally compact group of finite positive dimension contains a one-parameter subgroup.

K. Iwasawa.

Graev, M. I. Free topological groups. Amer. Math. Soc. Translation no. **35**, 61 pp. (1951).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 279–324 (1948); these Rev. **10**, 11.

Pettis, B. J. On continuity and openness of homomorphisms in topological groups. Ann. of Math. (2) **52**, 293–308 (1950).

In this article the author gives necessary and sufficient conditions for a homomorphism $h: X\to Y$ between two topological groups to be continuous or open. After noticing that $A^{-1}A$ is a neighborhood of the identity of a topological group whenever A is a second category Baire set, it is shown that h is continuous and X of second category if and only if $h^{-1}(V)$ contains a second category Baire set for any neighborhood V of the identity of Y. Dually, h is open and Y of second category if and only if $h(U)$ contains a second category Baire set for every neighborhood U of the identity of X. Assuming moreover that X and Y are metrizable, then, in case X is complete, h is open and has a closed kernel if and only if its graph is closed and it maps neighborhoods of the identity in X onto somewhere dense sets. Dually, in case Y is complete, h is continuous if and only if it has a closed graph and h^{-1} maps neighborhoods of the identity of Y onto somewhere dense sets. These facts extend well-known theorems of Banach and Freudenthal. [The recent extensions of the type considered by J. Dieudonné and L. Schwartz [Ann. Inst. Fourier Grenoble **1** (1949), 61–101 (1950); these Rev. **12**, 417] and G. Köthe [Math. Z. **53**, 203–209 (1950); these Rev. **12**, 417] are not explicitly subsumed by the author's propositions. The author pointed out to the reviewer that at the end of theorem 18 the phrase "and X is of second category" should be added.]

L. Nachbin (Rio de Janeiro).

Buck, R. Creighton. Generalized group algebras. Proc. Nat. Acad. Sci. U. S. A. **36**, 747–749 (1950).

Soit G un groupe topologique, C l'espace de Banach des fonctions complexes uniformément continues dans G, avec la norme $\|f\|=\sup_{x\in G}|f(x)|$, $B(C)$ l'algèbre de Banach des opérateurs linéaires continus dans C. L'auteur associe à G la sous-algèbre $\mathfrak{B}$ de $B(C)$ commutante des translations à gauche $f\to f_b$ $(f_b(x)=f(bx))$. On vérifie aisément que pour que $f\in C$ soit un caractère de G, il faut et il suffit que $T(f)=\lambda(T)f$ pour tout $T\in\mathfrak{B}$, et $T\to\lambda(T)$ est un homomorphisme continu de $\mathfrak{B}$ dans le corps K des nombres complexes, dont f est la restriction à G (identifié à son image par la représentation régulière droite de G dans $B(C)$). Mais inversement la restriction à G d'un tel homomorphisme de $\mathfrak{B}$ n'est pas nécessairement un caractère de G pour la topologie initiale de G. L'auteur donne des conditions pour qu'il en soit ainsi pour les restrictions à G des homomorphismes de certaines sousalgèbres de $\mathfrak{B}$. Il montre enfin que $\mathfrak{B}$ est isomorphe au dual $\mathfrak{C}$ de C d'une façon canonique, la structure d'algèbre ainsi définie sur $\mathfrak{C}$ étant celle définie par le produit de composition lorsque G est compact.

J. Dieudonné (Baltimore, Md.).

NUMBER THEORY

Rodeja F., E. G.-. On a question of Euler functions of consecutive orders. Revista Mat. Hisp.-Amer. (4) **10**, 95–98 (1950). (Spanish)

Let $\phi(n)$ be Euler's totient function and let $\phi_1(n)=\phi(n)$, $\phi_2(n)=\phi(\phi(n))$, $\cdots$ be the successive iterates of ϕ. The question mentioned in the title is that of discovering integers n such that (1) $n=\phi_1(n)+\phi_2(n)+\cdots+\phi_m(n)$, where $\phi_m(n)=1$ and $\phi_{m-1}(n)>1$. Examples of such integers are $n=3^a$ and certain other numbers such as 15, 39, 111, 327, 8751. The author proves the following theorem. Let $F_r=2^{2^r}+1$ be a Fermat prime and let $Q(r, k, p)$ be a prime of the form 2^kF_rp+1. Let $n(r, k, p, m, s)=F^mQ^s$ satisfy condition (1). Then $r=0$, $k=2$, $m=1$, and $s=1$ so that $n=3(4\cdot 3^p+1)$, the latter factor being a prime. This generalizes a result of L. Perez Cacho [Revista Mat. Hisp.-Amer. (3) **1**, 45–50 (1939); these Rev. **1**, 290] who considered only $n(0, k, p, m, 1)$. *D. H. Lehmer* (Berkeley, Calif.).

Ortega Costa, Joaquin. The explicit formula for the prime number function $\pi(x)$. Revista Mat. Hisp.-Amer. (4) **10**, 72–76 (1950). (Spanish)

The author suggests that the prime number function $\pi(x)$ be studied by means of the function $\delta(n)=\prod_{j,k=1}^{n-1}(jk-n)$. He derives some formulas of the following type:

$$\pi(n)=n-(2\pi)^{-1}\int_0^{2\pi}\sum_1^n \cos\left(x\delta(m)\right)dx.$$

P. Scherk (Saskatoon, Sask.).

★**Neville, E. H. The Farey Series of Order 1025, Displaying Solutions of the Diophantine Equation $bx-ay=1$.** Royal Society Mathematical Tables, Vol. I. Cambridge, at the University Press, 1950. xxix+405 pp. (1 plate). $18.50.

This volume is the first of a new series of British tables sponsored by the Royal Society. The main table gives the numerators and denominators of all reduced rational numbers P_n/Q_n with $0<P_n<Q_n\leqq 1025$ arranged in increasing order. The table contains $1+\sum_{n=1}^{1025}\phi(n)=319765$ such fractions and is nearly 100 times as extensive as previous tables. The sequence of fractions is "folded back on itself" at $\frac{1}{2}$, the numerators of P_n and Q_n-P_n being given in the same place with Q_n. No approximate decimal equivalents are given, but due to the remarkably uniform distribution of these values throughout most of the table, it is not difficult to locate by page and line number the position of a given fraction, especially if use is made of the two page "decimal index." The introduction illustrates the application of the table to such problems as the rational approximation to irrationals and the solution of linear Diophantine equations. There are four appendices. (1) The Farey series of order 50. Here decimal equivalents are also given to 5D. (2) The Farey series of order 64 on a single page. (3) The denominators of the Farey series of order 100. (4) A factor table of the first 1049 integers on both sides of a loose card.

D. H. Lehmer (Berkeley, Calif.).

Dénes, Peter. Über die Unlösbarkeit der Diophantischen Gleichung $x^{np}+y^{np}=p^m\cdot z^{np}$ in ganzen Zahlen x, y, z, m, n, wenn p eine reguläre Primzahl ist und $p>3$. Monatsh. Math. **54**, 175–182 (1950).

The equation in the title is shown to have no nonzero integer solutions x, y, z, if m and n are positive integers; also, if $p=3$ and $m\not\equiv 2 \bmod 3$. More generally, in the real subfield $k(\zeta+\zeta^{-1})$, if E_1, E_2, and E_3 are units and ξ, χ, ψ are prime to the ideal $[\Lambda]$, $\Lambda=(1-\zeta)(1-\zeta^{-1})$, and s is an integer greater than p, then the equation $E_1\xi^p+E_2\chi^p+E_3\Lambda^s\psi^p=0$ is impossible. *G. Pall* (Chicago, Ill.).

Birman, Abraham. Proof and examples that the equation of Fermat's last theorem is solvable in integral quaternions. Riveon Lematematika **4**, 62–64 (1950). (Hebrew. English summary)

An infinitude of solutions of $x^n+y^n=z^n$ in Lipschitz integral quaternions is constructed for each positive integer $n=1$ or 5 (mod 6) and for $n=3$, x and y being conjugate quaternions in each solution. *I. Niven* (Eugene, Ore.).

Auluck, F. C., Singwi, K. S., and Agarwala, B. K. On a new type of partition. Proc. Nat. Inst. Sci. India **16**, 147–156 (1950).

Using the standard method of steepest descent, applied to an integral of the generating functions round a suitable contour, asymptotic formulae are derived for the number of partitions of a large integer n into parts satisfying various conditions. A typical result is the following: If $\omega(n)$ is the number of partitions of n into integers, each of which may occur only an odd number of times, then $\omega(n)\sim[\gamma/(2\pi n)]\exp(2\gamma n^{\frac{1}{2}})$, with

$$\gamma^2=(\pi^2/12)+\int_0^1\log(1+y-y^2)(dy/y).$$

F. J. Dyson (Birmingham).

Shapiro, Harold N. Some remarks on a theorem of Erdös concerning asymptotic density. Proc. Amer. Math. Soc. **1**, 590–592 (1950).

The author gives a simple proof and a slight extension of a theorem of the reviewer on the asymptotic density of the sum of two sequences [Ann. of Math. (2) **43**, 65–68 (1942); these Rev. **3**, 165]. *P. Erdös* (Aberdeen).

Lepson, Benjamin. Certain best possible results in the theory of Schnirelmann density. Proc. Amer. Math. Soc. **1**, 592–594 (1950).

Let $0\leqq\alpha\leqq 1$, $0\leqq\beta\leqq 1$ be arbitrary real numbers. The author constructs two sequences of densities α and β whose sum has density $\rho=\min(1, \alpha+\beta)$. Thus the author shows that Mann's theorem [Ann. of Math. (2) **43**, 523–527 (1942); these Rev. **4**, 35] is best possible for every α and β.

P. Erdös (Aberdeen).

LeVeque, W. J. On representations as a sum of consecutive integers. Canadian J. Math. **2**, 399–405 (1950).

If m and s are positive integers, then $\gamma_1(m, s)$ denotes the number of representations

$$m=r+(r+s)+(r+2s)+\cdots+(r+(k-1)s);$$

and $\gamma(m, s)$ denotes the number of solutions under the restriction $r>0$. It is shown that $\gamma_1(m, s)$ equals the number of divisors of m if s is even, and twice the number of odd divisors of m if s is odd. The author determines the average order of $\gamma_1(m, s)$ and of $\gamma(m, s)$. [In the last line of theorem 5 the factor $\frac{1}{2}$ should be omitted. The proof of theorem 6 can be simplified considerably by the remark that $\sum_{m=1}^{n}\gamma(m, s)$ equals the number of lattice points (x, y) satisfying $x>0$, $y>0$, $xy\leqq 2n$, $y>(x-1)s$, $y\equiv(x-1)s \pmod 2$.]

N. G. de Bruijn (Delft).

Davenport, H. Sums of three positive cubes. J. London Math. Soc. **25**, 339–343 (1950).

Let $R(N)$ denote the number of positive integers $\leqq N$ that are representable as the sum of three positive cubes. It is proved that $R(N)>N^{\alpha-\epsilon}$ for $\epsilon>0$ and $N>C(\epsilon)$, with $\alpha=47/54$. This is an improvement of a previous result ($\alpha=13/15$) of the author [C. R. Acad. Sci. Paris **207**, 1366–1368 (1938)]. *N. G. de Bruijn* (Delft).

Davis, C. S. The minimum of an indefinite binary quadratic form. Quart. J. Math., Oxford Ser. (2) **1**, 241–242 (1950).

If m is the lower bound of $|f(x, y)|$ for integers x, y not both zero, where f is a real, binary quadratic form of positive discriminant d, and $k=d/m^2$, a simple proof is given that no forms exist with $0<k<5$, $5<k<8$, or $12<k<13$. Replacing f by an equivalent form $\pm m(x-\alpha_1 y)(x-\alpha_2 y)/(1-\epsilon)$, the proof is based on a discussion of integers which may lie between the roots of $(x-\alpha_1)(x-\alpha_2)=1-\epsilon$ and $-1+\epsilon$. Compare Pall [Math. Mag. **21**, 255 (1948); these Rev. **10**, 182], Cassels [Ann. of Math. (2) **50**, 676–685 (1949); these Rev. **11**, 643], and Ollerenshaw [J. London Math. Soc. **23**, 148–153 (1948); these Rev. **10**, 236] for other simple proofs. *G. Pall* (Chicago, Ill.).

Steuerwald, Rudolf. Über die Perioden regelmässiger Kettenbrüche für Quadratwurzeln aus ganzen Zahlen. Math. Z. **52**, 686–697 (1950).

This paper is concerned with a generalization of theorems "on the phenomenon of greatest middle in the cycle of a class of periodic continued fractions" of Muir [Proc. Roy. Soc. Edinburgh. Sect. A. **12**, 578–592 (1884); cf. Perron, Die Lehre von den Kettenbrüchen, Teubner, Leipzig and Berlin, 2d ed., 1929, § 27, pp. 110–115]. Using the notation of Perron's book, the author formulates the following problem. If r, Q, and c are integers and $r\geqq 1$, $Q\geqq 1$, $0\leqq c\leqq Q-1$, to determine positive integers b_p, $p=0, 1, \cdots, r$, and D, such that $\sqrt{D}$ has the regular continued fraction expansion

$$\sqrt{D}=[b_0, \overline{b_1, \cdots, b_{r-1}, b_r, b_{r-1}, \cdots, b_1, 2b_0}],$$

and

$$2A_{r-1}=Q(b_rB_{r-1}+2B_{r-2}), \quad 2DB_{r-1}=Q(b_rA_{r-1}+2A_{r-2}),$$

$$cA_{r-2,1}+B_{r-2,1}=Qa_{r-3,1}. \tag{1}$$

The cases $c=0$, $Q=2$ and $c=1$, $Q=2$ are those treated by Muir [loc. cit.]. Explicit formulas for b_0, D, and b_r in terms of positive integer solutions $b_1, b_2, \cdots, b_{r-1}$ of (1) and an integer m (subject to stated conditions) are given. There are examples and theorems on properties of solutions, and a classification of the solutions in the classical case $Q=2$. *H. S. Wall* (Austin, Tex.).

Schogt, C. A theorem of Petr from number theory. Math. Centrum, Amsterdam. Rapport **ZW-1950-009**, 10 pp. (1950). (Dutch)

A new proof is given of the following theorem of Petr [Časopis Pěst. Mat. Fys. **56**, 57–66 (1927)]: If D is a positive integer, not a square, then there is exactly one triple $(D_1, D_2, e)\neq(1, D, 1)$, with $D_1D_2=D$, $D_1<D_2$, $e=\pm 1$ or ± 2, for which there are integers u, v such that $D_1u^2-D_2v^2=e$ and $(u, D_2)=(v, D_1)=1$. Petr's proof used continued fractions; the present proof uses only properties of the units in the quadratic field $k(\sqrt{D})$. *W. J. LeVeque.*

Yamashita, Chitose. On the computations of the indices of the group of norm residues and of the group of power residues without employing logarithm. Tôhoku Math. J. (2) **1**, 279–284 (1950).

This contains a computation of the local norm index for ramified cyclic extensions of algebraic number fields. The basic method is to apply Herbrand's lemma to certain multiplicative subgroups defined by an additive normal basis. Although the author does not mention this, his proof is valid also for p-adic completions of fields of algebraic functions of one variable over a Galois field; but the computations are a bit long. The proof for power residues has only a trivial generalization to function fields. *G. Whaples* (Bloomington, Ind.).

Čudakov, N. G., and Rodosskiĭ, K. A. On generalized characters. Doklady Akad. Nauk SSSR (N.S.) **73**, 1137–1139 (1950). (Russian)

The authors define a generalized character as a complex-valued function $h(n)$ on the positive integers such that $h(mn)=h(m)h(n)$, $|h(n)|$ is 0 or 1 for any n, and $\sum_{n=1}^{x}h(n)$ is bounded for all positive integers x. The nonprincipal residue-characters of Dirichlet [cf. E. Landau, Vorlesungen über Zahlentheorie, Hirzel, Leipzig, 1927, vol. 1, pp. 83–87] obviously satisfy these requirements. By considering the sum $\sum_{n=1}^{m}(m-n)^2\sum_{d\mid n}h(d)$ with a suitable m, the authors prove that if $h(n)$ is a real generalized character and if M is the maximum of $|\sum_{n=1}^{x}h(n)|$ for all positive integers x, then $\sum_{n=1}^{\infty}h(n)n^{-1}>(14M)^{-1}$. They also remark that the lemma used by Estermann [J. London Math. Soc. **23**, 275–279 (1948); these Rev. **10**, 356] in his proof of Siegel's theorem is valid for real generalized characters. *P. T. Bateman* (Urbana, Ill.).

Čudakov, N. G., and Linnik, Yu. V. On a class of completely multiplicative functions. Doklady Akad. Nauk SSSR (N.S.) **74**, 193–196 (1950). (Russian)

A generalized character $h(n)$ in the sense of the paper reviewed above is completely determined by the values of $h(p)$, where p runs over the primes. The set of primes p for which $h(p)\neq 0$ is called the basis of $h(n)$. It is easy to construct generalized characters for which the basis consists of exactly one prime. This paper is devoted to proving that there are no generalized characters with a finite basis consisting of more than one prime. The proof uses quantitative work of Vinogradov on uniform distribution [cf. Trav. Inst. Math. Stekloff **23** (1947); these Rev. **10**, 599] and work of Gelfond [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **1939**, 509–518; these Rev. **1**, 295] on the rational approximations to $\log\alpha/\log\beta$, where α and β are positive algebraic numbers such that $\log\alpha/\log\beta$ is irrational (and thus transcendental). It is unknown whether or not there exist any generalized characters with an infinite basis other than the nonprincipal Dirichlet characters. *P. T. Bateman* (Urbana, Ill.).

Ankeny, N. C., and Chowla, S. The relation between the class number and the distribution of primes. Proc. Amer. Math. Soc. **1**, 775–776 (1950).

Let $d\equiv 3 \pmod 4$ and a prime; $h(d)$ the class number of the field $R((-d)^{\frac{1}{2}})$, where R is the field of rational numbers; $\pi(x; d, l)$ the number of primes not greater than x and congruent to $l \pmod d$; r a positive integer; $A<2$; and $c=(1-\frac{1}{2}A)/25$. It is shown that if $\pi(x;d,l)<Ax/((d-1)\log x)$

for any s such that $d<s<e^{(d)^{1/r}}$ and all t which are non-residues (mod d), then $h(d)>r$ for $d>N=N(A,r)$.

W. H. Mills (New Haven, Conn.).

Petersson, Hans. Zwei Bemerkungen über die Weierstrasspunkte der Kongruenzgruppen. Arch. Math. 2, 246–250 (1950).

Let $\Gamma(N)$ be the principal congruence subgroup of Stufe N of the modular group with genus p and with τ_0 a point in a fundamental region $\mathfrak{F}$. Let $M(\tau_0)$ be the set of all automorphic functions belonging to $\Gamma(N)$ which are regular everywhere in $\mathfrak{F}$ except at τ_0. Let $\lambda_i(\tau_0)$ be the ith gap in the sequence of possible pole-orders of the functions in $M(\tau_0)$ and let $\beta(\tau_0)=\beta_N(\tau_0)=\sum_{i=1}^{p}(\lambda_i(\tau_0)-i)$. If $\beta(\tau_0)>0$ the point τ_0 is called a Weierstrass-point of $\Gamma(N)$. By counting in two ways the number of such points in $\mathfrak{F}$ the author gives a rather simple proof that, for $N=1, 2, \cdots$, none of the algebraic forms defined by $\Gamma(N)$ is hyperelliptic. If $A=\beta_N(\infty)$, then by a similar type of argument it is shown that if $N\geqq 7$ we have $6A\equiv 0 \pmod N$ except in the following five cases (q denotes a prime): (a) N not square-free; (b) $N=q\geqq 7$; (c) $N=2q$, $q\geqq 5$, $q\neq 7, 11$; (d) $N=3q$, $q\geqq 5$; (e) $N=6q$, $q\geqq 5$, $q\neq 7$.

T. M. Apostol (Pasadena, Calif.).

Davenport, H., and Rogers, C. A. Diophantine inequalities with an infinity of solutions. Philos. Trans. Roy. Soc. London. Ser. A. 242, 311–344 (1950).

Let $F(x_1, \cdots, x_n)$ be a continuous real function of n real variables satisfying $F(tx_1, \cdots, tx_n)=t^hF(x_1, \cdots, x_n)$ for all real t, where h is a positive integer. Let $x_1, \cdots, x_n$ denote any real forms in $u_1, \cdots, u_n$ of determinant 1. Now if the inequality $|F(x_1, \cdots, x_n)|\leqq\lambda$ has at least one solution in integers $(u_1, \cdots, u_n)\neq(0, \cdots, 0)$ one may ask whether for each $\lambda'>\lambda$ the inequality (1) $|F(x_1, \cdots, x_n)|\leqq\lambda'$ has infinitely many such solutions. Carrying further Mahler's work on star bodies [Proc. Roy. Soc. London. Ser. A. 187, 151–187 (1946); Nederl. Akad. Wetensch., Proc. 49, 331–343, 444–454, 524–532, 622–631 (1946); these Rev. 8, 195, 12], which enables one to prove under certain general conditions that the answer to the above question is affirmative, the authors prove a number of theorems concerning inequalities of the type (1) having an infinity of integer solutions.

The inequality (2) $|F(x_1, \cdots, x_n)|\leqq 1$ defines a starbody K in R_n. This body is called automorphic, if F possesses automorphisms, i.e., linear transformations with real coefficients which leave $|F|$ invariant, such that every point $(x_1, \cdots, x_n)$ of the body K can be transformed by an appropriate automorphism into a point of a bounded region. The body is called fully automorphic if moreover every point except the origin can be transformed by appropriate automorphisms into points arbitrarily far from the origin.

Now consider a lattice defined by n linear forms $x_1, \cdots, x_n$ of the integer variables $u_1, \cdots, u_n$ with determinant $\Delta>0$. This lattice is said to be admissible for K if none of its points except the origin is strictly inside K. The lower bound of the determinants Δ of all lattices which are admissible for K is denoted by $\Delta(K)$. Further, the definitions of critical lattices for K, reducible starbodies, and boundedly reducible starbodies are used [the reader can find these in the reviews of the cited papers by Mahler]. Moreover, they give a new definition: K is said to be fully reducible if there exists a bounded starbody H contained in K such that $\Delta(H)=\Delta(K)$, whereas H has exactly the same critical lattices as K. If K is fully reducible, it is certainly boundedly reducible. The authors then prove theorem 1. Suppose K is fully automorphic and suppose Λ is a lattice with determinant $\Delta>0$. Then (a) if $\Delta<\Delta(K)$, there are an infinity of solutions of (2) with strict inequality; (b) if K is boundedly reducible and $\Delta\leqq\Delta(K)$, there are an infinity of solutions of (2); (c) if K is fully reducible and $\Delta\leqq\Delta(K)$, there are an infinity of solutions of (2) with strict inequality, unless Λ is a critical lattice of K, in which case there are an infinity of solutions of (2) with equality.

The authors prove the fully-reducibility for the following special starbodies; calculating the corresponding values of $\Delta(K)$ they apply theorem 1 to them: $|x_1x_2|\leqq 1$ $(\Delta(K)=\sqrt{5})$; $-1\leqq x_1x_2\leqq k$ $(\Delta(K)=(k^2+4k)^{\frac{1}{2}}$ for any positive integer $k)$; $|x_1x_2x_3|\leqq 1$ $(\Delta(K)=7)$; $|(x_1^3+x_2^3)x_3|\leqq 1$ $(\Delta(K)=\frac{1}{2}\sqrt{23})$; $|x_1^3+x_2^3-x_3^3|\leqq 1$ $(\Delta(K)=\sqrt{\frac{4}{3}})$; $|x_1^3+x_2^3+x_3^3-x_4^3|\leqq 1$ $(\Delta(K)=\sqrt{\frac{4}{7}})$; $|x_1^3+x_2^3-x_3^3-x_4^3|\leqq 1$ $(\Delta(K)=\frac{4}{3})$.

Further, the authors investigate the possible distributions of the lattice points within given bodies. A starbody K_0 contained in K is said to generate K if for every bounded part K' of K there is an automorphism Ω of K, such that ΩK_0 contains K'. If K_0 generates K, then obviously ΩK_0 also generates K for every automorphism Ω of K. Now the authors prove: theorem 4. Suppose K_0 generates K, K being a fully automorphic starbody. Suppose Λ is a lattice with determinant Δ. Then (a) if $\Delta<\Delta(K)$ there are an infinity of points of Λ strictly inside K_0; (b) if K is boundedly reducible, and $\Delta\leqq\Delta(K)$, there are an infinity of points of Λ in K_0; (c) if K is fully reducible and $\Delta\leqq\Delta(K)$, then there are an infinity of points of Λ strictly inside of K_0, unless Λ is a critical lattice of K, in which case there are an infinity of points of Λ on the boundary of K_0. Theorem 8. Suppose K_0 generates K, K not necessarily being fully automorphic. Assume that Θ is an automorphism of K which reduces K_0. Suppose that Λ is a lattice with determinant Δ which has no point other than the origin in $C(K_0, \Theta)$. Then the conclusions (a), (b), (c) of theorem 4 are valid. Remark. The set $C(K_0, \Theta)$ is defined to consist of all points which are common to all starbodies $\Theta^m K_0$ $(m=1, 2, \cdots)$. The authors give various applications of the above theorems which cannot be explicitly quoted here, and moreover deduce "isolation theorems": If for a class of functions F the value λ_0 is the greatest lower bound of all λ' for which (1) has infinitely many solutions for all functions F of the class, one may ask which "critical" functions F may be excluded in order to improve the inequality (1) for the remaining functions of the class [cf. Markoff's results for binary quadratic forms, Math. Ann. 15, 381–407 (1879)]. Thus, e.g., new results are deduced for the case $F(x, y, z)=(x^2+y^2)z$, where x, y, z denote linear forms in u, v, w, of determinant 1. The paper closes with an interesting conjecture concerning the simultaneous approximation of two real numbers θ_1, θ_2 by rationals p_1/q, p_2/q.

J. F. Koksma (Amsterdam).

Koksma, J. F. Some theorems on Diophantine inequalities. Math. Centrum Amsterdam, Scriptum no. 5, i+51 pp. (1950).

The author proves the following theorem. Let Q denote the m-dimensional parallelepiped $a_\mu\leqq x_\mu<b_\mu$ $(\mu=1, \cdots, m)$, where $a_1, \cdots, a_m$, $b_1, \cdots, b_m$ are integers. Let $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ be $2n$ real numbers satisfying $\alpha_\nu<\beta_\nu\leqq\alpha_\nu+1$ $(\nu=1, \cdots, n)$, and let $f_1(x), \cdots, f_n(x)$ be n real functions defined for all lattice points $(x)=(x_1, \cdots, x_m)$ of Q. Let $\lambda_1, \cdots, \lambda_n$ be n positive numbers $\geqq 1$ and let

$$M_\nu=\lambda_\nu \log \{e \min (n, \lambda_\nu)\},$$

$\nu=1, \cdots, n$. Let $N(Q)$ denote the number of lattice points

$(x)\varepsilon Q$ and let $T(Q)$ be defined by $T(Q)N(Q)$

$$=\sum_{(h)}^{*}\Big|\sum_{(x)\varepsilon Q}\exp\{2\pi i(h_1f_1(x)+\cdots+h_nf_n(x)\}\Big|\prod_{\nu=1}^{n}p_{h_\nu,\nu},$$

where

$$p_{h_\nu,\nu}=\min\{\beta_\nu-\alpha_\nu+75/\lambda_\nu,\ 1-(\beta_\nu-\alpha_\nu)+75/\lambda_\nu,\ 30/|h_\nu|\}$$

for $h_\nu\neq 0$, $\nu=1,\cdots,n$, $p_{0,\nu}=\beta_\nu-\alpha_\nu+75/\lambda_\nu$, $\nu=1,\cdots,n$, and $\sum_{(h)}^{*}$ is extended over all nonzero lattice points $(h)=(h_1,\cdots,h_n)$ in n-dimensional space such that $|h_\nu|\leqq M_\nu$, $\nu=1,2,\cdots,n$. Then the number $N^*(Q)$ of solutions of the Diophantine system $\alpha_\nu\leqq f_\nu(x)<\beta_\nu$ (mod 1), $\nu=1,\cdots,n$, satisfies the inequality

$$|N^*(Q)/N(Q)-\prod_{\nu=1}^{n}(\beta_\nu-\alpha_\nu)|$$

$$\leqq\{\prod_{\nu=1}^{n}(\beta_\nu-\alpha_\nu+75/\lambda_\nu)-\prod_{\nu=1}^{n}(\beta_\nu-\alpha_\nu)\}+T(Q).$$

This result is an improvement of a theorem said to have been obtained by the author and van der Corput in 1935 but never published [the one-dimensional case is given in the author's Diophantische Approximationen, Springer, Berlin, 1936, Kapitel IX, Satz 4]. The author also gives two corollaries of his main theorem, namely (1) an improvement of an unpublished theorem of van der Corput [quoted without proof, loc. cit., Kapitel X, Satz 2] and (2) a refinement of a theorem of Erdös and Turán [Nederl. Akad. Wetensch., Proc. 51, 1146–1154, 1262–1269 = Indagationes Math. 10, 370–378, 406–413 (1948); these Rev. 10, 372, last theorem quoted in the review]. *P. T. Bateman* (Urbana, Ill.).

Chabauty, Claude. Sur la répartition modulo un de certaines suites p-adiques. C. R. Acad. Sci. Paris **231**, 465–466 (1950).

Soit p un nombre premier ($\neq 2$ pour simplifier). Soit Q_p l'ensemble des nombres p-adiques. Tout $\alpha\varepsilon Q_p$ est de la forme $\sum_{h=r}^{+\infty}a_hp^h$, r entier rationnel, a_h pris dans l'ensemble $0,\pm 1,\cdots,\pm\frac{1}{2}(p-1)$. On pose $[\alpha]=\sum_{(h\leqq 0)}a_hp^h$ (c'est un nombre rationnel), et $\{\alpha\}=\sum_{(h>0)}a_hp^h$ (c'est un nombre p-adique de valeur absolue <1 qu'on appelera la reste modulo un p-adique de α). L'auteur énonce avec de brèves indications sur les démonstrations, plusieurs théorèmes analogues à ceux de Pisot et le rapporteur sur la répartition modulo un au sens ordinaire. Il montre que parmi certains ensembles de nombres algébriques absolus, $\theta\varepsilon Q_p$, définis par des inégalités sur les valeurs absolus p-adiques et ordinaires de θ et de ses conjugués absolus, l'un peut être caractérisé par la décroissance géométrique de $|\{\lambda\theta^n\}|_p$ (λ nombre convenable, n entier rationnel tendant vers $+\infty$), l'autre par la convergence de $\sum[\lambda\theta^n]^s$ (convergence au sens réel, $[\ ]$ ayant la définition p-adique donnée ci-dessus), et que ce dernier ensemble est fermé et admet des ensembles dérivées non vides de tout ordre entier (pour la topologie p-adique). *R. Salem* (Paris).

ANALYSIS

Gill, John P. Elementary concepts of functional means and dispersions. Math. Mag. **24**, 65–75 (1950).

Knaster, B. Sur une équivalence pour les fonctions. Colloquium Math. **2**, 1–4 (1949).

Aczel has shown [Bull. Amer. Math. Soc. **54**, 392–400 (1948); these Rev. **9**, 501] that if $M(x,y)$ is (i) increasing, (ii) continuous, (iii) bisymmetric: $M[M(x,y),M(z,u)]=M[M(x,z),M(y,u)]$, (iv) reflexive: $M(x,x)=x$, and (v) symmetric: $M(x,y)=M(y,x)$, then there exists a continuous and increasing function $f(x)$ such that

$$(*)\qquad M(x,y)=f^{-1}\left[\frac{f(x)+f(y)}{2}\right].$$

Ryll-Nardzewski has shown [Studia Math. **11**, 31–37 (1949); these Rev. **12**, 12] that (*) holds provided $M(x,y)$ is (1) increasing, (2) continuous, and (3) distributive in itself: $M[x,M(y,z)]=M[M(x,y),M(z,x)]$. Since the converse of each of the above results is valid, it follows that the conditions (i)–(v) and (1)–(3) are equivalent. The author now writes the conditions as (iii) $[(x,y)(z,u)]=[(x,z)(y,u)]$, and so on, and shows in an entirely formal way that if (1) holds, then conditions (iii)–(v) are equivalent to (3). *E. F. Beckenbach* (Los Angeles, Calif.).

Tornheim, Leonard. On n-parameter families of functions and associated convex functions. Trans. Amer. Math. Soc. **69**, 457–467 (1950).

The author develops the theory of convex functions generalized in the following way. Let F be an n-parameter family of single-valued real continuous functions $f(x)$ on an interval $a\leqq x\leqq b$ such that for every set of points (x_i,y_i), $i=1,\cdots,n$, with $a\leqq x_1<\cdots<x_n\leqq b$ there is exactly one $f(x)$ with $f(x_i)=y_i$; then $g(x)$ is said to be convex with respect to F if $g(x)$ is real-valued and continuous on (a,b) and its graph intersects the graph of no member of F more than n times. It is shown, for instance, that if a function $g(x)$ is convex with respect to F, and its graph meets the graph of a member f of F at n points, then the graphs of the two functions cross at each of the n points. Further, if $n\geqq 3$ and all the functions of F have derivatives on $a<x<b$, then so does any function g which is convex with respect to F. The best approximant in F of a continuous function g is a function f^* in F for which

$$\max_{a\leqq x\leqq b}|f^*(x)-g(x)|=\inf_{f\varepsilon F}\max_{a\leqq x\leqq b}|f(x)-g(x)|.$$

It is shown that if g is continuous, then there is exactly one best approximant in F of g. *E. F. Beckenbach.*

Ulčar, Jože. Ein Grenzwertsatz. Bull. Soc. Math. Phys. Macédoine **1**, 41–45 (1950). (Macedonian. German summary)

Let $f(x,\alpha)$ and $g(x,\alpha)$ be real-valued continuous functions in the rectangle $a\leqq x\leqq b$, $c\leqq\alpha\leqq d$. Suppose that for some α_0, $\lim_{\alpha\to\alpha_0}f(x,\alpha)/g(x,\alpha)=k$ and that $\int_a^b g(x,\alpha)\,dx\neq 0$ for $|\alpha-\alpha_0|$ sufficiently small. Then

$$\lim_{\alpha\to\alpha_0}\int_a^b f(x,\alpha)\,dx\Big/\int_a^b g(x,\alpha)\,dx=k.$$

The proof given for this elementary fact is incorrect. Something is also stated about the case $\int_a^b g(x,\alpha)\,dx=0$ for $|\alpha-\alpha_0|$ arbitrarily small, but no proof is given and the matter is left quite obscure. *E. Hewitt.*

Pul'kin, S. P. Oscillating sequences of iterations. Doklady Akad. Nauk SSSR (N.S.) **73**, 1129–1132 (1950). (Russian)

The author investigates oscillatory behaviour of a sequence $X=X(x_0)=\{x_n\}$ generated by x_0 under the iteration

$x_n = f(x_{n-1})$, $n = 1, 2, \cdots$, where $f(x)$ is a real-valued function in $-\infty < x < \infty$, $f(x)$ has a finite number of extrema, its graph a finite number of points of inflexion, and $f''(x)$ is continuous in stretches. When x_n is periodic with smallest period p the set $\{x_0, x_1, \cdots, x_{p-1}\}$ is called a cycle of order p: when $p = 1$, x_0 is a fixed point of f. When the qth derivative $X^{(q)}$ of $X = \{x_n\}$, considered as a set of points, is a finite set, X is said to be of the qth class; when, further, $X^{(q)}$ consists of the points of a finite system of "unstable" cycles, X is said to oscillate relative to this system. The author's main theorem (too complicated for reproduction here) gives a sufficient condition for such oscillation and for the existence of iteration sequences of the higher classes. He indicates the proof for the special case of a system of two second order cycles. He observes that his sufficient conditions are fulfilled by quite simple functions, for example, $f(x) = 4x - x^2$, which therefore yields sequences that oscillate relative to the two unstable fixed points $x = 0$ and $x = 3$.

H. P. Mulholland (Bath).

Thijsen, W. P. On iterated exponentiation. Simon Stevin **27**, 177–192 (1950). (Dutch)

The following theorem is proved. Let $\varphi(u)$ be a single-valued continuous function in (α, β), taking values in (α, β), and such that $\varphi(u) - u$ has no zeros inside (α, β). Let $\varphi_1(u) = \varphi(u)$, $\varphi_{n+1}(u) = \varphi(\varphi_n(u))$. Then $\varphi_n(u) \to \alpha$ or β as $n \to \infty$, according as $\varphi(u) < u$ or $\varphi(u) > u$. The author then generalizes this to the case where $\varphi(u) - u$ has a finite number of changes of sign, and applies it to $\varphi(u) = x^u$. For $x > 1$ the situation is simple because $\varphi(u)$ increases; for $x < 1$ it is necessary to consider instead the increasing function $\varphi(\varphi(u))$; the odd and even iterates behave differently. A lengthy calculation is required to discuss the roots of $x^{x^u} = u$. For $0 < x < e^{-e}$ there are three roots σ, λ, τ; for $e^{-e} < x < 1$, only one, λ; the author gives a brief table of the roots. It follows that for $0 < x < e^{-e}$ either $\varphi_{2m}(u) \to \sigma$ and $\varphi_{2m+1}(u) \to \tau$ or vice versa; otherwise $\varphi_n(u) \to \lambda$ or ∞; the possibilities, as well as whether the convergence is monotone increasing or decreasing, are displayed in a figure. Define ${}^0x = 1$, ${}^1x = x$, ${}^2x = x^x$, ${}^{n+1}x = (x \text{ with exponent } {}^nx)$; then putting $u = 0$ and 1 leads to the result that ${}^{2m+1}x \to \sigma$, ${}^{2m}x \to \tau$ for $0 < x < e^{-e}$, where $\sigma = x^\tau$, $\tau = x^\sigma$, $\sigma < \tau$; while ${}^nx \to \lambda$ for $e^{-e} \leqq x \leqq e^{1/e}$, where $\lambda = x^\lambda$; and ${}^nx \to \infty$ for $x > e^{1/e}$ [D. F. Barrow, Amer. Math. Monthly **43**, 150–160 (1936)]. The author also studies the case $\varphi(u) = (ax)^{(x^u)}$.

R. P. Boas, Jr. (Evanston, Ill.).

Loewner, Charles. Some classes of functions defined by difference or differential inequalities. Bull. Amer. Math. Soc. **56**, 308–319 (1950).

The author shows that his former study [Math. Z. **38**, 177–216 (1934)] of functions which he called monotonic of order n can be subordinated to the consideration of more general classes of functions which have distinguished group theoretical properties. A class S of continuous and strictly increasing real functions $f(x)$ of a real variable, defined in open intervals, is called a transformation subgroup provided that (a) if $f(x)$, defined on (a, b), is in S, then $f(x)$ considered only on a subinterval $(a', b') \subset (a, b)$ is in S; (b) if $f(x) \in S$ and $g(x) \in S$, and the range of $f(x)$ falls into the domain of $g(x)$, then $g[f(x)] \in S$; and (c) the class S satisfies a closure condition relative to uniformly convergent sequences of functions of S. If, in addition, the inverse of each function in S is also in S, then S is said to be a transformation group. The strictly monotonically increasing functions of fixed order n form a semigroup satisfying the additional condition that S contains the group of proper projective transformations, and it is with these semigroups that the author is especially concerned. After reviewing properties of monotonic functions of order n, the author outlines results concerning the above transformation semigroups in their relation to differential inequalities; proofs of the results announced in this invited address are deferred, to be given in another paper.

E. F. Beckenbach.

Tagamlickiĭ, Ya. A. On functions whose derivatives satisfy certain inequalities. Doklady Akad. Nauk SSSR (N.S.) **75**, 337–340 (1950). (Russian)

The author shows, by using the Abel interpolation series, that the conditions

$$(*) \qquad |f^{(k)}(x)| \leqq (-1)^k [-xe^{-x}]^{(k)} = (k - x)e^{-x}$$

for $0 \leqq x \leqq k$; $k = 0, 1, 2, \cdots$, imply $f(x) \equiv Cxe^{-x}$. A function satisfying (*) is easily seen to be an entire function such that $|f(z)| \leqq |z| e^{|z|}$ and $f^{(k)}(k) = 0$; thus the author's result may be thought of as an extension of the fact that $|f(z)| \leqq e^{\lambda|z|}$, $\lambda < 1$, and $f^{(k)}(k) = 0$ imply $f(z) \equiv 0$.

R. P. Boas, Jr. (Evanston, Ill.).

Meyers, Leroy F., and Sard, Arthur. Best interpolation formulas. J. Math. Physics **29**, 198–206 (1950).

Les auteurs continuent l'étude des meilleures formules d'interpolation [Amer. J. Math. **71**, 80–91 (1949); ces Rev. **10**, 576]. Soient les valeurs $x(0)$, $x(1)$, $\cdots$, $x(m)$ d'une fonction $x(t)$ et u une valeur fixe de la variable. Si $A = \sum_{i=0}^{m} a_i(u)x(i)$ est égal à $x(u)$ pour tous les polynomes de degré n, on peut écrire $x(u) - A = \int_K x^{(n+1)}(t)k(t, u)\,dt$, l'intégrale étant étendue au plus petit intervalle K contenant u et celles des valeurs $0, 1, \cdots, m$ qui correspondent à des a_i non nuls. La "meilleure" formule est celle qui minimise $M^2 = |K| \int_K k^2(t, u)\,dt$. Le présent mémoire étudie les cas $n = 1$, $m = 1, 2, 3, 4$. La meilleure formule est l'interpolation linéaire entre les points les plus voisins $n = 2$, $m = 3, 4, 5$. La meilleure formule ne correspond pas à une interpolation quadratique. On donne effectivement les meilleures formules pour $n = 2$, $m = 3, 4$ et les valeurs de M^2 pour $n = 2$, $m = 3, 4, 5$.

J. Kuntzmann (Grenoble).

Ostrowski, Alexandre. Sur la variation de la matrice inverse d'une matrice donnée. C. R. Acad. Sci. Paris **231**, 1019–1021 (1950).

Let $\{\gamma_{\mu,\nu}\}$ be the inverse of a matrix $\{C_{\mu,\nu}\}$ whose elements depend on a parameter t. Let $\Delta = (\sum |\gamma_{\mu,\nu}|^2)^{\frac{1}{2}}$, $C'_{\mu,\nu}$ the derivative with respect to t of $C_{\mu,\nu}$. Using a generalization of the Schwarz inequality the author shows that the derivative of Δ^{-1} does not exceed $(\sum |C'_{\mu,\nu}|^2)^{\frac{1}{2}}$.

F. J. Murray (New York, N. Y.).

Ribeiro de Albuquerque, J. The Wronskian determinants. Gaz. Mat., Lisboa **11**, nos. 44–45, 4–7 (1950). (Portuguese)

Let Δ_n be a determinant whose elements are functions of a function of x which are defined continuous and admitting derivatives on the interval (a, b). Let Δ_p represent any p-rowed minor of Δ_n whose rows are consecutive. The author shows that Δ_n is the Wronskian of the functions in its first row if and only if the derivative Δ_p' of every Δ_p is obtained from Δ_p by differentiating the last row of Δ_p. Conditions for the vanishing of Δ_n (not necessarily a Wronskian) are also considered.

A. A. Albert.

Calculus

***Landau, Edmund. Differential and Integral Calculus.** Translated by Melvin Hausner and Martin Davis. Chelsea Publishing Co., New York, N. Y., 1951. vii+366 pp.

English translation of the author's Einführung in die Differentialrechnung und Integralrechnung, Noordhoff, Groningen, 1934.

***Appell, Paul. Analyse mathématique à l'usage des candidats au certificat de mathématiques générales et aux grandes écoles. Tome I. Analyse des courbes, surfaces et fonctions usuelles, intégrales simples.** 6th ed. Gauthier-Villars, Paris, 1951. ix + pp. 1–408. 2,000 francs.

***Appell, Paul. Analyse mathématique à l'usage des candidats au certificat de mathématiques générales et aux grandes écoles. Tome II. Équations différentielles, développements en séries, nombres complexes, intégrales multiples, probabilités, déterminants, exercices.** 6th ed. Gauthier-Villars, Paris, 1951. Pp. 409–850. 2,200 francs.

The 5th edition of these two texts appeared in 1937 and 1938, respectively. This edition, like the last, was prepared by Georges Valiron.

***Duschek, Adalbert. Vorlesungen über höhere Mathematik. Band II. Unendliche Reihen. Integration und Differentiation der Funktionen von mehreren Veränderlichen. Abschluss der Wahrscheinlichkeitsrechnung. Fehlertheorie und Ausgleichsrechnung. Lineare Algebra. Tensorfelder.** Springer-Verlag, Wien, 1950. vi+386 pp. $7.80; bound, $8.70.

Unendliche Reihen; Funktionen von mehreren Veränderlichen; die Integration von Funktionen mehrerer Veränderlicher; die Fundamentalsätze der Wahrscheinlichkeitsrechnung, Fehlertheorie und Ausgleichsrechnung; Lineare Algebra; Tensoranalysis. *Table of Contents.*

Nagumo, Mitio. Note on the derivative of an exponential function. J. Osaka Inst. Sci. Tech. Part I. 1, 121 (1949). (Esperanto)

Fine, N. J. Proof of a theorem of Jacobi. Proc. Amer. Math. Soc. 1, 666–667 (1950).

The transformation

$$\int_0^\pi G(\cos x)\cos nx\,dx = \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)}\int_0^\pi G^{(n)}(\cos x)\sin^{2n}x\,dx,$$

given by Jacobi [J. Reine Angew. Math. 15, 1–26 (1836)], is proved without using properties of special functions.

G. Crane (Chicago, Ill.).

Hochrainer, A. Ebene Tensoren und komplexe Zahlen. Österreich. Ing.-Arch. 4, 222–234 (1950).

The representation of points and lines in the plane by means of complex quantities is shown to be identical with their representation by means of plane tensors of the second rank in whose matrix the elements of the principal diagonal are equal and of the other diagonal equal but of opposite signs. This explains why multiplication of complex numbers differs from multiplication of vectors.

L. M. Milne-Thomson (Greenwich).

Tzénoff, Iv. Vecteurs libres et glissants et leur application dans la géométrie des systèmes de droites. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 37, 523–580 (1941); 38, 1–48 (1942). (Bulgarian. French summary)

Theory of Sets, Theory of Functions of Real Variables

Sierpinski, Waclaw. Dernières recherches et problèmes de la théorie des ensembles. Matematiche, Catania 5, 3–14 (1950).

Lecture summarizing results in the theory of sets obtained in Poland during recent years.

Kurepa, Georges. La condition de Souslin et une propriété caractéristique des nombres réels. C. R. Acad. Sci. Paris 231, 1113–1114 (1950).

Let E be a totally ordered class, continuous and endless. The following result is announced: In order that E should be ordinally similar to the class of real numbers it is necessary and sufficient that the Cartesian product E^2 (or indeed, E^n for any $n>1$) should satisfy the condition of Souslin. *J. Todd* (Los Angeles, Calif.).

***Borel, Émile. Les paradoxes de l'infini.** Gallimard, Paris, 1946. 237 pp.

Lévy, Paul. Les paradoxes de l'infini et le calcul des probabilités. Bull. Sci. Math. (2) 73, 186–192 (1949).

If the axiom of choice legitimizes choosing a point in each linear set A, and if the notion of choosing a point in A is necessarily identified with the probability of A, it might appear [see the preceding title] that the axiom of choice implies that the definition of linear Lebesgue measure can be extended to all linear sets A, and that the extension will obey the usual laws. The author denies the validity of this reasoning. *J. L. Doob* (Ithaca, N. Y.).

***Landau, Edmund. Foundations of Analysis. The Arithmetic of Whole, Rational, Irrational and Complex Numbers.** Translated by F. Steinhardt. Chelsea Publishing Company, New York, N. Y., 1951. xiv+134 pp. $3.25.

Translation of Grundlagen der Analysis, Akademische Verlagsgesellschaft, Leipzig, 1930.

***Vitali, G., e Sansone, G. Moderna teoria delle funzioni di variabile reale. Parte I. Aggregati, analisi delle funzioni, integrazione, derivazione.** 3d ed. Nicola Zanichelli, Bologna, 1951. vii+222 pp. 3,000 Lire.

Reprint with minor changes of the 2d edition [1943; these Rev. 7, 376].

Gleason, A. M. A note on a theorem of Helson. Colloquium Math. 2, 5–6 (1949).

Let G be a set and $\mathfrak{G}$ the collection of all subsets of G. Consider a binary operation $\circ$ on $\mathfrak{G}$ such that: (1) $\mathfrak{G}$ forms a group under $\circ$; and (2) $A\circ B\subset A\cup B$ for all A and B in $\mathfrak{G}$. It is shown that these two properties characterize the set operation of symmetric difference over $\mathfrak{G}$. This characterization strengthens, by dropping two additional conditions on $\circ$, a previous result of Helson [Colloquium Math. 1, 203–205 (1948); these Rev. 10, 518]. *W. Gustin.*

Sikorski, R. Independent fields and Cartesian products. Studia Math. **11**, 171–184 (1950).

The principal purpose of this paper is to establish the equivalence of the theory of (countably) independent collections of Boolean (o, −) algebras and the theory of Cartesian products. On the way the author presents a detailed discussion of the pertinent facts concerning the extendability of Boolean homomorphisms. The essential idea (i.e., a proof of Banach's extension theorem by the introduction of a suitable product space) is the same as that given by S. Sherman [Amer. J. Math. **72**, 612–614 (1950); these Rev. **12**, 15]. The reception dates of the two papers differ by two days. *P. R. Halmos* (Chicago, Ill.).

Marczewski, E., and Sikorski, R. Remarks on measure and category. Colloquium Math. **2**, 13–19 (1949).

The purpose of this paper is to present various conditions on a Borel measure in a topological space sufficient to ensure the existence of (*) a decomposition of the space into a set of measure zero and a set of the first category. The authors give a simple proof of the fact that (*) exists for Hausdorff measures in separable metric spaces. A cardinal number m is said to have measure zero if every finite measure defined on the class of all subsets of a set of power m vanishes identically whenever it vanishes on all finite sets. If a metric space contains a dense subset whose power has measure zero, then (*) exists for every σ-finite measure that vanishes on all finite sets. For nonmetric spaces this result is false: there exists a finite Borel measure μ in a compact Hausdorff space such that $\mu(X)=0$ if and only if X is of the first category. *P. R. Halmos* (Chicago, Ill.).

Fodor, G. On two problems concerning the theory of binary relations. Publ. Math. Debrecen **1**, 199–200 (1950).

Suppose that to each point x of the unit interval there corresponds a set $S(x)$, called the picture of x, such that $g(x)=d(S(x), x)>0$. Two points are called independent if neither of them belongs to the picture of the other; it is convenient also to call a set independent if any two points of it are independent. The author proves that (a) there exists an independent set of the power of the continuum and (b) if there exists a measurable function f such that $0<f(x)\leqq g(x)$ for all x, then there exists an independent set of positive measure. [Reviewer's comment. The author's proof proves the stronger assertion (c) if μ is a nonnegative, countably additive measure defined on a Boolean σ-algebra B containing all intervals, if μ is not identically zero, and if there exists a function f measurable with respect to B and such that $0<f(x)\leqq g(x)$ for all x, then there exists an independent set E in B such that $\mu(E)>0$. If B is the class of Lebesgue measurable sets and μ is Lebesgue measure, then (c) reduces to (b); if B is the class of all sets and μ is the measure that is ∞ on all sets of the power of the continuum and 0 on all sets of smaller cardinality, then (c) reduces to (a). Not having made this comment, the author has to present his proof twice.] *P. R. Halmos.*

Denjoy, Arnaud. La régularité métrique des familles d'ensembles. C. R. Acad. Sci. Paris **231**, 1013–1015 (1950).

Denjoy, Arnaud. Métrique des ensembles et des fonctions. C. R. Acad. Sci. Paris **232**, 195–197 (1951).

These notes continue the discussion of a generalized Vitali theorem [same C. R. **231**, 560–562, 600–601, 737–740 (1950); these Rev. **12**, 246, 324]. The theorems mentioned at the close of the last abstract utilize in their hypotheses a notion of "perfect regularity" for the set of sets G. This concept is altered in these notes by the introduction of kernel and envelope sets. The set F is a kernel subset of $\Delta(G)$ if the set $\Delta(G(F))-F$ is of φ-measure zero, where $G(F)$ is the totality of points of F covered indefinitely by the sets γ of G. The set O is an envelope set if $\Delta(G)-O$ is a kernel set. Then G is said to be perfectly regular if for every φ-measurable subset H of $\Delta(G)$ and for every $\epsilon>0$ there exists an envelope set O containing H and a kernel set F contained in H such that $\varphi(O-H)<\epsilon$ and $\varphi(H-F)<\epsilon$. For such perfectly regular sets the metric density theorem as well as the theorem paralleling the relation between Lebesgue integrals and absolutely continuous set functions holds. While the kernel and envelope sets reduce to closed and open sets in Euclidean space, they do not in general give rise to a topology. The concluding section of the second note shows how a proper selection from and dilation of the nonmeasurable sets due to Vitali make it possible to find on the unit interval a denumerable number of disjoint nonmeasurable sets, each of outer measure unity.

T. H. Hildebrandt (Ann Arbor, Mich.).

Berge, Claude. Sur l'isovalence et la régularité des transformateurs. C. R. Acad. Sci. Paris **231**, 1404–1406 (1950).

The author considers set valued functions A on an additive family $D_A=\{e\}$ of sets such that $A(\phi)=\phi$, the empty set, and $A(e_1\cup e_2)=Ae_1\cup Ae_2$. The function A is called regular if $Ae_1=Ae_2$ implies $e_1=e_2$, isovalent if $A(e_1\cap e_2)=Ae_1\cap Ae_2$. If $\{f\}$ is a set of (real valued) functions on a space X, an operator $f\to g=\bar{A}f$ is called isovalent of $f(X)=(\bar{A}f)(X)$ for all f. A one-to-one correspondence between isovalent functions A and isovalent operators $\bar{A}$ is established; further it is shown that isovalence of A and the proposition that $Ae=\phi$ implies $e=\phi$ implies its regularity, while regularity and the proposition that D_A is multiplicative implies isovalence. *G. K. Kalisch* (Minneapolis, Minn.).

Berge, Claude. Sur l'inversion des transformateurs. C. R. Acad. Sci. Paris **232**, 134–136 (1951).

This is a continuation of a preceding note [see the preceding review]. Using the notations introduced there, the author considers the inverse function A^{-1} and establishes several of the usual properties. Also, if A is completely additive and regular, $e=AE$ (where $E\varepsilon\{Ae\}$) implies that $E=A^{-1}e$. The adjoint function A^* (defined on $\{Ae\}$) is defined by $A^*E=[y; E\cap A\{y\}\neq\phi]$, where $\{y\}$ is a one element set. The function A^* is completely additive and $A^{**}=A$ if A is completely additive; if A is regular and isovalent, so is A^*. *G. K. Kalisch.*

Berge, Claude. Sur une théorie ensembliste des jeux alternatifs. C. R. Acad. Sci. Paris **232**, 294–296 (1951).

This is a sketchy description of "alternative" 2-person games [no reference is made to von Neumann and Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1st ed., 1944, 2d ed., 1947; these Rev. **6**, 235; **9**, 50], using terms defined in two of his earlier papers [see the two preceding reviews]. The two players, (X_1) and (X_2), play alternately; the central concept is that of "position" e which is a description of the game at the time of a certain move with an indication of who is to play next. Games are then classified in terms of the functions Γ, Γ^*, and Γ^{-1}, where Γe is the set of positions following e. It is shown that if Γ is isovalent it is regular, and if it is regular there may be no tie. Finally there is a discussion of "good"

and "bad" positions and moves (where one or the other player must win or loose, respectively). *G. K. Kalisch.*

Halperin, Israel. Discontinuous functions with the Darboux property. Amer. Math. Monthly **57**, 539–540 (1950).

Using a Hamel basis or the well-ordering theorem the author establishes the existence of functions $f_1(x)$ and $f_2(x)$ which take on every real number as value $\aleph$ times on every interval or on every perfect set, respectively. Both functions $f_1(x)$ and $f_2(x)$ are non-Lebesgue measurable on every measurable set of positive measure. *A. Rosenthal.*

Ellis, H. W. Examples of integrals that are discontinuous in sets of positive measure. Trans. Roy. Soc. Canada. Sect. III. (3) **44**, 37–42 (1950).

This note gives an instance of a function $F(x)$ on $a \leqq x \leqq b$ whose set of discontinuities is not of zero measure but which is an indefinite integral of a function $f(x)$ according to certain generalized integral definitions. The integrals in question are the AP-integral of Burkill [Math. Z. **34**, 270–278 (1931)] in which approximate continuity and approximate derivatives are used in the Perron definition; the β-integral of Ridder [Fund. Math. **22**, 136–162 (1934)] in which the indefinite integral $F(x)$ is approximately continuous and generalized absolutely continuous and the approximate derivative of $F(x)$ is $f(x)$ almost everywhere; and an integral of the author [Canadian J. Math. **1**, 113–124 (1949); these Rev. **10**, 520] which is similar to the β-integral except that approximate continuity is replaced by Cesàro continuity ($F(x)$ is special Denjoy integrable, and $\lim_{h\to 0} h^{-1}\int_{x_0}^{x_0+h} F(t)\,dt = F(x_0)$). The example centers in a nondense perfect set which is not of zero measure.

T. H. Hildebrandt (Ann Arbor, Mich.).

See the note in Errata, on p. 1002.

Grosswald, Emil. Functions of bounded variation. Duke Math. J. **17**, 313–315 (1950).

Let $f(x)$ be a function of bounded variation for $a \leqq x \leqq b$, let I be a subdivision $(a = x_0 < x_1 < \cdots < x_n = b)$ of the interval $[a, b]$, and let $S(I) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$, so that the total variation V of $f(x)$ on $[a, b]$ is the supremum of $S(I)$ as I varies. Further, let c_i $(i = 1, 2, \cdots)$ be the abscissae in $[a, b]$ at which $f(x)$ has a positive external saltus

$$s_i = |f(c_i+0) - f(c_i)| + |f(c_i) - f(c_i-0)| - |f(c_i+0) - f(c_i-0)|,$$

arranged so that $s_1 \geqq s_2 \geqq \cdots$. The author proves that for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ and an $N(\epsilon) < \infty$ such that $S(I) > V - \epsilon$ whenever I is such that (i) $|x_i - x_{i-1}| < \delta(\epsilon)$, $i = 1, 2, \cdots, n$, and (ii) the points $x_0, x_1, \cdots, x_n$ of subdivision include $c_1, c_2, \cdots, c_N$, with $N > N(\epsilon)$. This theorem would be false without the condition (ii).

H. P. Mulholland (Bath).

Fubini, Guido. Il teorema di riduzione per gli integrali doppi. Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 125–133 (1950).

Italian translation of a lecture given at Princeton University in 1942.

Love, E. R. Repeated integrals involving Cauchy principal values. J. London Math. Soc. **25**, 184–189 (1950).

Let $f(x, y)$ be integrable over the finite square $a < x < b$, $a < y < b$. The author establishes sufficient conditions for the equality

$$(1) \qquad \int_a^b dx \int_a^b \frac{f(x,y)}{y-x}\,dy = \int_a^b dy \int_a^b \frac{f(x,y)}{y-x}\,dx,$$

where the inner integrals are Cauchy principal values, defined, for example, by

$$(2) \qquad \int_a^b \frac{f(x,y)}{y-x}\,dy = \lim_{h\to 0}\left(\int_a^{x-h} + \int_{x+h}^b\right)\frac{f(x,y)}{y-x}\,dy,$$

where the limit exists almost everywhere. Another theorem gives conditions for the equality (1) when $a = -\infty$ and $b = +\infty$. In this case the inner integrals are Cauchy principal values in the sense of mean convergence, that is, the limit on the right of (2) is replaced by a limit in the mean. The theory of Cauchy principal values for Riemann type integrals was developed extensively by G. H. Hardy [Proc. London Math. Soc. (1) **34**, 16–40, 55–91 (1901); (1) **35**, 81–107 (1902); (2) **7**, 181–208 (1908)]. The author gives modern versions of two of Hardy's theorems, the latter's differentiability conditions near the line of singularities being replaced by a uniform Lipschitz condition in one case and a type of integrated Lipschitz condition in the other.

G. B. Price (Lawrence, Kan.).

Theory of Functions of Complex Variables

Leja, Franciszek. Les problèmes de la théorie des fonctions analytiques dans les travaux récents. Časopis Pěst. Mat. Fys. **74** (1949), 79–88 (1950). (Polish. French summary)

A review of papers on analytic functions which were reviewed in these Rev. during 1948.

Vicente Gonçalves, J. La limite de Walsh. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1**, 187–188 (1950).

This paper contains a proof of the theorem of Walsh [Ann. of Math. (2) **25**, 285–286 (1924)] which states that the zeros of $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ satisfy the inequality $|x| \leqq |a_1| + |a_2|^{\frac{1}{2}} + \cdots + |a_n|^{1/n}$. Other upper bounds for the moduli of the zeros are also stated. *E. Frank.*

Iglesias Garrido, Tomas. Introduction to the development of the theory of analytic functions defined as limits of sequences of polynomials. Revista Mat. Hisp.-Amer. (4) **10**, 99–143 (1950). (Spanish)

The author develops a suggestion of Ríos [same Revista (4) **3**, 310–311 (1943); these Rev. **6**, 205] that the theory of functions can be developed by defining a function as analytic if it is a uniform limit of polynomials. In the author's exposition the sequence of ideas is the following: normal families; Taylor series; differentiation; analytic continuation; singular points. *R. P. Boas, Jr.*

Shah, S. M. On the coefficients of an entire series of finite order. J. London Math. Soc. **26**, 45–46 (1951).

For an entire function $f(z) = \sum a_n z^n$ of positive finite order ρ, there are two definitions of a "lower type," namely, $l = \liminf r^{-\rho} \log M(r)$ and $\theta = \liminf (\rho e)^{-1} n |a_n|^{\rho/n}$. The author shows that $l \geqq \theta$, with equality if a_n/a_{n+1} is non-decreasing. *R. P. Boas, Jr.* (Evanston, Ill.).

Milloux, Henri. Sur les fonctions entières d'ordre fini ou nul. C. R. Acad. Sci. Paris **232**, 296–297 (1951).

Announcement of the following results for integral functions of finite positive order or of order 0 and sufficiently fast rate of growth. Theorem 1. Every Borel direction of $f(z)$ is also a Borel direction of its successive integrals.

Theorem 2. If L is a limiting direction of Borel directions of an infinity of different successive derivatives of $f(z)$, then L is a common Borel direction of $f(z)$, its successive derivatives and its successive integrals. Theorem 3. If $f(z)$ is of order $\rho>\frac{1}{2}$, the common Borel directions of $f(z)$, its derivatives and its successive integrals cannot be contained in an angle of opening less than $2\pi-\pi/\rho$ $(\rho<1)$ or π/ρ $(\rho>1)$.

W. H. J. Fuchs (Ithaca, N. Y.).

Inoue, Masao. Sur le module minimum des fonctions sousharmoniques et des fonctions entières d'ordre $<1/2$. Mem. Fac. Sci. Kyūsyū Univ. A. **4**, 183–193 (1949).

A number of theorems concerning the minimum modulus of continuous subharmonic functions are established. The following is typical. Let U be continuous, subharmonic, nonnegative, and unbounded in the finite plane. Let

$$M(r)=\max_{|z|=r} U,\quad m(r)=\min_{|z|=r} U,\quad \bar{\rho}=\limsup_{r\to\infty}\log M(r)/\log r,$$
$$\underline{\rho}=\liminf_{r\to\infty}\log M(r)/\log r.$$

Let $\bar{\mu}E$ and $\underline{\mu}E$ denote the upper and lower logarithmic density of a set E of positive numbers. Let $f(r)$ be positive and increasing in r and satisfy

$$\lim_{r\to\infty} f(r)=\infty,\quad \limsup_{r\to\infty}\log f(r)/\log r<\infty.$$

Let $\liminf_{r\to\infty}\log M(r)/\log f(r)=\lambda<\infty$. Then if $\bar{\rho}<\frac{1}{2}$,

$$\underline{\mu}E[m(r)>f(r)^{\lambda-\epsilon}]\geqq 1-2\bar{\rho};$$

if $\rho<\frac{1}{2}$, $\bar{\mu}E[m(r)>f(r)^{\lambda-\epsilon}]\geqq 1-2\rho$. Use is made of inequalities given by Beurling [Thesis, Uppsala, 1933].

M. Heins (Providence, R. I.).

Inoue, Masao. A note on the minimum modulus of integral functions of lower order $<1/2$. Math. Japonicae **2**, 41–47 (1950).

Let c denote a positive number, $S=\{a_n\}$ a monotone increasing sequence of positive numbers with $\lim a_n=+\infty$, $E_\rho^c(S)$ the set of positive x satisfying $|x-a_n|<ca_n^{1-\rho}$ for at least one n. The author shows: (1) If f is a non-constant entire function of order ρ, type τ, where $0<\rho<\frac{1}{2}(1-2c\tau\rho^2)$, and if the upper logarithmic density of $E_\rho^c(S)$ is one, then

$$\limsup_{n\to\infty}\{(2^{\frac{1+2\rho'}{1-2\rho'}}-1)\log m(a_n)-\log M(a_n)\}=+\infty$$

for $\rho/(1-2c\tau\rho^2)<\rho'<\frac{1}{2}$. Here $M(r)$ is the maximum modulus and $m(r)$ is the minimum modulus of f on $|z|=r$. (2) If f is an entire function of order ρ $(0<\rho<+\infty)$, type τ, and lower order $\lambda<\frac{1}{2}(1-2c\tau\rho^2)$, and if $E_\rho^c(S)$ is of logarithmic density one, then for $\lambda/(1-2c\tau\rho^2)<\lambda'<\frac{1}{2}$,

$$\limsup_{n\to\infty}\{(2^{\frac{1+2\lambda'}{1-2\lambda'}}-1)\log m(a_n)-\log M(a_n)\}=+\infty.$$

Related theorems and applications are given. Use is made of previous results of the author [see the preceding review].

M. Heins (Providence, R. I.).

Hiong, King-Lai. Sur une extension du second théorème fondamental de R. Nevanlinna. C. R. Acad. Sci. Paris **230**, 1635–1636 (1950).

Let ϕ, ψ be assigned functions. In the notation of the Nevanlinna theory the author announces under suitable conditions the inequality

$$[1-o(1)]T[r,f]<N[r,f]+N[r,(f-\psi)^{-1}]+N[r,(f'-\psi)^{-1}]+S(r).$$

To the reviewer this appears to be an immediate consequence of the results of Milloux [e.g., Les fonctions méromorphes . . . , Actualités Sci. Ind. no. 888, Hermann, Paris, 1940; these Rev. **7**, 427] which the author quotes.

W. K. Hayman (Exeter).

Hiong, King-Lai. Sur les fonctions méromorphes et leurs dérivées. C. R. Acad. Sci. Paris **231**, 323–325 (1950).

Let f be meromorphic in the finite plane and let φ be entire and satisfy $T(r,\varphi)=o\{T(r,f)\}$. The defect of φ for f, $\delta(\varphi,f)$ is defined as $1-\limsup N(r,(f-\varphi)^{-1})/T(r,f)$ and φ is said to be exceptional for f when $\delta(\varphi,f)>0$. A number of theorems whose proofs are based on a previous result of the author [see the preceding review] are announced. (I) If f is meromorphic of finite order λ and such that $\delta(\infty)=1$ and if f' admits as an exceptional function of defect 1 an entire function $\psi_1\not\equiv 0$ such that $T(r,\psi_1)=o[T(r,f')]$, then $N(r,(f-\varphi)^{-1})$ belongs to the same type as $T(r,f)$ for every entire function φ satisfying (γ) $T(r,\varphi)=o[T(r,f)]$ and $\varphi'\not\equiv\psi_1$. (II) Under the hypotheses of (I), if f belongs to the divergence class, then with r_ν denoting the moduli of the zeros of $f-\varphi$, $\sum r_\nu^{-\lambda}$ diverges for every φ satisfying (γ). (III) Under the hypotheses of (II), if λ is an integer, the genus of $f-\varphi$ is λ for every entire function φ satisfying (γ). Other related theorems are stated.

M. Heins.

Łojasiewicz, S. Une démonstration du théorème de Fatou. Ann. Soc. Polon. Math. **22** (1949), 241–244 (1950).

A simple proof is given of the well-known theorem that a function regular and bounded inside a rectifiable contour possesses angular limits almost everywhere.

W. K. Hayman (Exeter).

Jenkins, James A. Positive quadratic differentials in triply-connected domains. Ann. of Math. (2) **53**, 1–3 (1951).

Let D be a triply-connected domain, and suppose that $P(z)\,dz^2$ is regular inside D and nonnegative on the boundary. Then the loci $P(z)\,dz^2>0$ form closed curves in D.

W. K. Hayman (Exeter).

Jenkins, J. A., and Spencer, D. C. Hyperelliptic trajectories. Ann. of Math. (2) **53**, 4–35 (1951).

Let $R(w)=\prod_{n=1}^{N}(w-\alpha_n)^{p_n}$, where the p_n are positive or negative integers, $\zeta=\int\sqrt{R(w)}\,dw$. A curve γ, $R(w)\,dw^2<0$ or $\Re\zeta=$constant, $\Im\zeta$ monotonic, is called a hyperelliptic trajectory. The points α_n are called critical, finite if the order $p_n\geqq -1$, infinite otherwise. The order of $w=\infty$ is $-4-\sum p_n$. The following results are proved. As $\Im\zeta\to\infty$ a trajectory γ either (i) tends to a limit which is an infinite critical point; or (ii) has an end Λ which includes the whole of γ, in which case (a) γ describes the same Jordan curve infinitely often or (b) if α is a small arc, $\Im\zeta=$constant, containing a point of γ in its interior, the intersection of γ with α is a perfect linear set. The case (ii) (b) cannot occur if $R(w)$ has less than 4 distinct poles. It does occur for suitable elliptic integrals, $N=4$. The complement of γ and Λ consists of a finite number of simply connected domains, each containing at least one critical point in its interior, or is empty. If all trajectories are drawn through the finite critical points, the exterior set consists of a finite number of domains which are mapped by ζ onto a strip or half-

plane or by $\exp(\zeta/c)$ for a suitable c onto a circle or ring. It is shown that all four types of domain may occur for the same $R(w)$. *W. K. Hayman* (Exeter).

Hayman, W. K. Symmetrization in the theory of functions. Tech. Rep. no. 11, Navy Contract N6-ori-106 Task Order 5. Stanford University, Calif., 1950. 38 pp.

The author investigates a large number of applications to the theory of functions of two methods of symmetrization: the Steiner method, recently treated by Pólya and Szegö [Amer. J. Math. 67, 1–32 (1945); these Rev. 6, 227], and circular symmetrization, introduced by Pólya [C. R. Acad. Sci. Paris 230, 25–27 (1950); these Rev. 11, 435]. Let $w=f(z)=a_0+a_1z+\cdots$ be regular in $|z|<1$ and let the values assumed by $f(z)$ in $|z|<1$ lie in a region D_f of the w-plane. Denote by D_f^* the symmetrized region (Steiner or circular) of D_f. Consider now the class of all functions $f(z)$ satisfying the above conditions with fixed a_0 and such that the corresponding regions D_f^* all lie in a fixed simply connected region D_0, symmetric in the axis of reals, other than the whole finite plane. Then $|a_1|$ attains its maximum when $f(z)$ maps $|z|<1$ in a one-to-one conformal manner on D_0. This general principle enables the author to obtain sharp bounds for $|a_1|$ in terms of a_0 and various geometric conditions on D_f. An important result in this direction is the following: Let $R_f=R$ be the least upper bound of all $r>0$ for which the circle $|w|=r$ is wholly contained in D_f and if there is no such r, let $R_f=0$. Then $|a_1|\leqq 4(|a_0|+R)$, with equality if $w=f(z)$ maps the circle $|z|<1$ onto the w-plane slit along the negative real axis from $-R$ to $-\infty$. This result sharpens a theorem of Landau [Rend. Circ. Mat. Palermo 46, 347–348 (1922)]. Under the above conditions the author also obtains sharp upper bounds for $|f(z)|$ and $|f'(z)|$ on $|z|=r$ $(0\leqq r<1)$. A further development of these ideas enables the author to extend completely the classical Koebe-Bieberbach theory of schlicht functions to the class of mean p-valent functions, first introduced and studied by Biernacki [Bull. Sci. Math. (2) 70, 45–51, 51–76 (1946); these Rev. 8, 326]. The two central results in this direction are: (I) If $f(z)=a_0+a_1z+\cdots$ is mean p-valent in $|z|<1$ and $f(z)\neq 0$, then for $|z|=r$ $(0\leqq r<1)$, $|a_1|\leqq 4p|a_0|$, $|a_0|[(1-r)/(1+r)]^{2p}\leqq|f(z)|\leqq|a_0|[(1+r)/(1-r)]^{2p}$, $|f'(z)|\leqq 4|a_0|p(1+r)^{2p-1}/(1-r)^{2p+1}$. Equality holds for $f(z)=[(1+z)/(1-z)]^{2p}$. (II) If $f(z)=z^p+a_{p+1}z^{p+1}+\cdots$ is mean p-valent in $|z|<1$, where p is a positive integer, then for $|z|=r$ $(0\leqq r<1)$, $|a_{p+1}|\leqq 2p$, $r^p/(1+r)^{2p}\leqq|f(z)|\leqq r^p/(1-r)^{2p}$, $|f'(z)|\leqq pr^{p-1}(1+r)/(1-r)^{2p+1}$. Furthermore, $f(z)$ assumes every value w in $|w|<1/4^p$ precisely p times in $|z|<1$. All inequalities become equalities for $f(z)=z^p/(1-z)^{2p}$. All, except the first, of the results in (II) are new. An inequality is also found for the class of bounded functions in $|z|<1$. *W. Seidel* (Rochester, N. Y.).

Szegö, Gabor. Conformal mapping of the interior of an ellipse onto a circle. Amer. Math. Monthly 57, 474–478 (1950).

Let E be the ellipse with foci at ± 1 in the z-plane, and with semi-axes a and b; and let $f(z)=c_1z+c_2z^2+\cdots$ provide a schlicht conformal mapping of the interior of E onto the interior of the unit circle, with $c_1>0$. The function $\log[f(z)/z]$ can be expanded into a series of Tchebychev polynomials; by an adroit application of the method of undetermined coefficients, the author finds the coefficients in this series and deduces the formula of Schwarz.

G. Piranian (Ann Arbor, Mich.).

Shniad, H. On analytic maps of circles into convex regions. Amer. Math. Monthly 57, 473–474 (1950).

The author establishes the following result. Let a function $w=f(z)$ have the properties that it is analytic in the open unit circle and that the map of the unit circle is contained in a convex region R. Then the map of the circle $|z|\leqq r$ $(r<1)$ is contained in a convex region obtained by deforming R homothetically, with respect to the map of the origin, in the ratio $2r/(r+1)$. The ratio $2r/(r+1)$ is the best possible, even if the region R is assumed to be bounded.

G. Piranian (Ann Arbor, Mich.).

Jørgensen, Vilhelm. On conformal mapping on a surface of a sphere. Mat. Tidsskr. B. 1950, 131–137 (1950). (Danish)

Univalent conformal maps of spherical annuli into the sphere are considered. An area theorem is proved for such maps with the aid of the Schwarz inequality and the isoperimetric inequality for curves on a sphere. Among the applications is a proof of the classical Koebe-Faber theorem.

M. Heins (Providence, R. I.).

Kufarev, P. P. On conformal mapping of complementary regions. Doklady Akad. Nauk SSSR (N.S.) 73, 881–884 (1950). (Russian)

Let z_1 and z_2 denote two distinct fixed points in the circle $|z|<R$ and let Γ be an arbitrary Jordan arc which decomposes $|z|<R$ into two regions B_1 and B_2 such that z_1 lies in B_1 and z_2 in B_2. Denote by $z=f_k(w)$, $f_k(0)=0$, $f_k'(0)>0$, the function which maps the unit circle $|w|<1$ conformally on the region B_k $(k=1, 2)$. Setting $I(\Gamma; z_1, z_2)=f_1'(0)f_2'(0)$, it is required to find that arc Γ for which $I(\Gamma; z_1, z_2)$ attains its maximum. This problem was proposed and solved by M. A. Lavrent'ev [Trav. Inst. Math. Stekloff 5, 159–245 (1934)]. The maximum is attained if Γ is the non-Euclidean line in the circle $|z|<R$ which bisects orthogonally the non-Euclidean segment which connects the two points z_1 and z_2. The author solves this problem by the method of parametric representations, based on Löwner's differential equation. *W. Seidel* (Rochester, N. Y.).

Lokki, Olli. Über Existenzbeweise einiger mit Extremaleigenschaft versehenen analytischen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 76, 15 pp. (1950).

The author establishes the theorem of Koebe which states that a plane region bounded by p contours admits a one-to-one directly conformal map onto a circular disk less $p-1$ concentric circular arcs. The proof is based upon extremal considerations. The author considers in a bounded region B whose boundary consists of p analytic Jordan curves functions of the form $\log(z-z_0)+(z-z_0)\eta(z)$ ($z_0\varepsilon B$, η analytic in B) and introduces a modified Dirichlet integral for this class. The function minimizing the modified Dirichlet integral is the logarithm of a Koebe mapping function. Concurrently the existence of a Green's function is established.

M. Heins (Providence, R. I.).

Alenicyn, Ju. E. On bounded functions in multiply connected regions. Doklady Akad. Nauk SSSR (N.S.) 73, 245–248 (1950). (Russian)

Let D denote an n-tuply connected region in the z-plane, not containing $z=\infty$, with nondegenerate boundary continua. For any fixed point $z=a$ in D, let R_a denote the family of all functions $g(z, a)$ which are regular and single-valued in D and for which $g(a, a)=0$ and $|g(z, a)|<1$ in D.

It was shown by Ahlfors [Duke Math. J. 14, 1–11 (1947); these Rev. 9, 24] that for all functions $g(z, a)$ of the family R_a the maximum of $|g'(a, a)|$ is attained by a function which maps D in a one-to-one and conformal manner on a full n-sheeted unit circle, and this extremal function is unique to within a rotation. Denoting by $F(z, a)$ that extremal function for which $F'(a, a)>0$, the author obtains by elementary applications of Ahlfors' theorem upper bounds for $|g'(z)|$, where $g(z)$ is regular and single-valued in D and satisfies one of various additional conditions, such as boundedness. Assuming in addition that D contains the origin, denote by S_p, where p is any positive integer, the family of functions $f(z)$, regular and single-valued in D, such that $\log f(z)/z^p$ is also regular and single-valued in D. By application of the above estimates the author obtains very simple proofs for two related theorems, of which we quote only the first: If $f(z)\in S_p$ and $|\log f(z)/z^p|<M$ in D, then $\Re\{zf'(z)/f(z)\}>0$ in the largest circle about the origin, all of whose points satisfy the condition $|z|F'(z, z)<p/M$. The radius of this circle gives the best possible value of the radius of p-valence, as well as of the radius of star-likeness about the origin for the class of functions under consideration. Both this and the second theorem generalize results of Nehari [Amer. J. Math. 71, 845–852 (1949); these Rev. 11, 426]. *W. Seidel* (Rochester, N. Y.).

***Bergman, Stefan. The Kernel Function and Conformal Mapping.** Mathematical Surveys, No. 5. American Mathematical Society, New York, N. Y., 1950. vii+161 pp. $4.00.

This book is a survey of methods and results centering around orthogonal analytic functions and associated kernel functions defined for multiply-connected domains of the complex plane. The scalar product of two functions f, g over a domain B is defined to be $J_B(f, g)=\int_B f\bar{g}\,d\omega$, where $d\omega$ denotes an element of area. The single-valued analytic functions f on B with finite norms form a complete Hilbert space $L^2(B)$. If $f(z)$ is analytic, then $|f(z)|^2$ is sub-harmonic, that is, $|f(z_0)|^2\leqq(\pi r^2)^{-1}\int_{|z-z_0|<r}|f(z)|^2\,d\omega$. Hence $L^2(B)$ is a particularly simple space in which convergence in the sense of the norm implies point-wise convergence. Moreover, there exists a reproducing kernel $K(z, \bar{\zeta})$ satisfying $f(\zeta)=J_B(f(z), \overline{K(x, \bar{\zeta})})$ for each function of $L^2(B)$. The kernel $K=K_B$ is a fundamental domain functional which gives rise, in particular, to the Bergman metric $K_B(z, \bar{z})|dz|^2$ in terms of which B is a complete metric space. The first four chapters are devoted to the development of these concepts.

Let $l^2(B)$ be the subspace of $L^2(B)$ which is composed of those functions having single-valued integrals over B. From the point of view of Riemann surface theory, L^2 is to be regarded as the space of square-integrable differentials over B, l^2 as the subspace of differentials having vanishing periods. Let $\tilde{K}_B(z, \bar{\zeta})$ be the kernel of $l^2(B)$, and let $G(z, \zeta)$ and $N(z, \zeta)$ denote the Green's and Neumann's functions of B. In the fifth chapter the following formulas, due to Schiffer, are derived: $K_B(z, \bar{\zeta})=-(2/\pi)\partial^2 G(z, \zeta)/\partial z\partial\bar{\zeta}$, $\tilde{K}_B(z, \bar{\zeta})=-(2/\pi)\partial^2 N(z, \zeta)/\partial z\partial\bar{\zeta}$. Moreover, the harmonic measures of the boundary curves of B are expressed in terms of the Green's function and formulas connecting the Green's and Neumann's functions are derived. Thus the domain functionals are all essentially expressed in terms of the Green's and Neumann's functions. The chapter also contains a discussion of the space of single-valued harmonic functions in terms of the scalar product defined by the Dirichlet integral. Under suitable normalization the reproducing kernel of this space is $(2\pi)^{-1}\{N(z, \zeta)-G(z, \zeta)\}$.

Next, the conformal mappings of a multiply-connected domain onto the various standard canonical domains are constructed in terms of the Green's and Neumann's functions, the harmonic measures, and their periods. These mappings are then expressed in terms of the kernels K_B and $\tilde{K}_B$. Then the space of analytic functions f in B which are square integrable over the boundary of B is taken up together with the associated Szegö kernel $\hat{K}(z, \bar{\zeta})$. An application is made to an extremal problem for bounded analytic functions. The presentation follows lines given originally by Garabedian.

The eighth chapter is devoted to an exposition of Schiffer's method of varying the Green's function. Schiffer's variational method differs from Hadamard's in that the boundary variation is replaced by an interior variation which is applicable to domains with arbitrary boundaries. Several applications of Schiffer's method are given. The ninth and final chapter relating to analytic functions of one variable contains a proof of the existence of mappings onto canonical slit domains by a method based on the kernel which is due to Garabedian and Schiffer and to Lehto.

The tenth chapter discusses generalizations of the foregoing theory to solutions of partial differential equations of elliptic type in two real variables, and the final chapter discusses generalizations to analytic functions of two complex variables defined over subdomains of Euclidean space.

The book is written in an elementary style which avoids abstract concepts almost entirely, and much of it can be read by anyone having an elementary knowledge of complex variables.

D. C. Spencer (Princeton, N. J.).

Gahov, F. D. On a case of Riemann's boundary problem for systems of n pairs of functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 549–568 (1950). (Russian)

This is a detailed account of the results announced in an earlier paper [Doklady Akad. Nauk SSSR (N.S.) 67, 601–604 (1949); these Rev. 11, 169]. A somewhat more general case is considered insofar as the matrix $\mathfrak{A}^-$ is assumed to consist of boundary values of functions that are analytic in the finite part of S^- but may have poles at infinity. Correspondingly, the solutions are permitted to have poles at infinity. The author also considers the case where the determinant of the coefficient matrix has zeros on the boundary. The solution of the problem is based on a transformation of the boundary conditions to a canonical form, which is similar to the transformation of the basis of an algebraic field to normal form as used in the theory of algebraic functions.

M. Golomb (Lafayette, Ind.).

Parreau, Michel. Sur certaines classes de fonctions analytiques uniformes sur les surfaces de Riemann. C. R. Acad. Sci. Paris 231, 751–753 (1950).

Announcement of results concerning means of functions analytic on a Riemann surface and extension of the Nevanlinna theory of meromorphic functions to Riemann surfaces. The author shows that the Nevanlinna conjecture: $C_{AB}\subset C_{AD}$ is valid generally. This was proved for surfaces of genus zero by Ahlfors and Beurling [Acta Math. 83, 101–129 (1950); these Rev. 12, 171]. *M. Heins.*

Myrberg, P. J. Über die Existenz von beschränktartigen automorphen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 77, 8 pp. (1950).

A function meromorphic on a Riemann surface is said to be of bounded type provided that the logarithm of its modulus may be represented as the difference of two negative harmonic functions (with possible isolated logarithmic singularities). It is shown that Riemann surfaces of finite genus with positive boundary admit non-trivial meromorphic functions of bounded type. Use is made of the fact that meromorphic functions on a closed Riemann surface are $(1, n)$. *M. Heins* (Providence, R. I.).

Virtanen, K. I. Über die Existenz von beschränkten harmonischen Funktionen auf offenen Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 75, 8 pp. (1950).

The author shows that Riemann surfaces which admit non-trivial harmonic functions with finite Dirichlet integral also admit non-trivial bounded harmonic functions. A proof is also given of the known result that the Riemann surfaces with null boundary in the sense of Nevanlinna are precisely those which do not possess a Green's function.

M. Heins (Providence, R. I.).

Tornehave, Hans. Some remarks concerning analytic manifolds. Mat. Tidsskr. B. 1950, 36–37 (1950).

Consider n analytic functions

$$f_j(z_1, \cdots, z_{2n}) = g_j(x_1, \cdots, x_{2n}, y_1, \cdots, y_{2n}) + ih_j(x_1, \cdots, x_{2n}, y_1, \cdots, y_{2n})$$

of $2n$ complex variables and assume that a part of the manifold $f_j=0$ $(j=1, \cdots, n)$ admits a representation in the form $y_\mu=\varphi_\mu(x_1, \cdots, x_{2n})$, $\mu=1, \cdots, 2n$. By use of the Cauchy-Riemann equations the author obtains the explicit form of the matrix $(\partial y_\mu/\partial x_j)$, $\mu, j=1, \cdots, 2n$, and in particular he obtains the result that the characteristic polynomial of this matrix is $(\lambda^2+1)^n$. As the author indicates, these results furnish generalizations of known results for the case $n=1$ which corresponds to the case of two complex variables. *W. T. Martin* (Cambridge, Mass.).

Pfluger, Albert. Quelques théorèmes sur une classe de fonctions pseudo-analytiques. C. R. Acad. Sci. Paris 231, 1022–1023 (1950).

The author studies the class of complex-valued functions $w(z)$ such that the absolute value of the differential quotient dw/dz, which depends on $\arg dz$, has a maximum and a minimum whose quotient is bounded. For this class of functions extensions of Schwarz's lemma, the Phragmén-Lindelöf theorem, and Liouville's theorem are obtained. The chief tool is a comparison of length and area in a suitable metric by means of Schwarz's inequality.

P. R. Garabedian (Stanford University, Calif.).

Theory of Series

Rios, Sisto. Problems of the rearrangement of series. Collectanea Math. 2, 115–127 (1949). (Spanish)

A survey of old and new results on rearrangement of numerical series and series of functions. *R. P. Boas, Jr.*

Milošević, Kovina. Sur la somme d'une série. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Éd. Spéc. 3, 42 pp. (1950). (Serbo-Croatian. French summary)

The author derives a formula, involving Bernoulli numbers, for finite sums of finite products of special finite sums. This is used to obtain a list of about 180 identities such as

$$\sum_1^n k(k+3)(k+4)(k+8) = \tfrac{1}{5}n^5+\tfrac{17}{4}n^4+\tfrac{188}{6}n^3+\tfrac{242}{4}n^2+\tfrac{1772}{30}n.$$

R. P. Agnew (Ithaca, N. Y.).

Milošević, K. Sur une formule sommatoire. Bull. Soc. Math. Phys. Serbie 2, nos. 1–2, 69–74 (1950). (Serbo-Croatian. French summary)

In case $a_k=0$ and $d_k=1$ for each k, the number S_n defined by

$$S_n=\sum_{j=0}^{n-1}\prod_{k=1}^{p}(a_k+jd_k)$$

reduces to the sum $0^p+1^p+2^p+\cdots+(n-1)^p$ for which there is a familiar expression, a polynomial in n with coefficients involving Bernoulli numbers, obtainable by a straightforward application of the Euler (or Euler-Maclaurin) summation formula. This paper treats the general case, obtaining for S_n an expression which is a polynomial in n with coefficients too complicated for elucidation in a short review. *R. P. Agnew* (Ithaca, N. Y.).

Markovitch, D. Sur quelques limites du module d'une somme. Bull. Soc. Math. Phys. Serbie 2, nos. 1–2, 31–35 (1950). (Serbo-Croatian. French summary)

If $a_0, a_1, \cdots, a_n$ are positive numbers, $0<\theta<\pi$, and $r=\max(a_{k+1}/a_k)$, then

$$\sum_{k=0}^{n}a_ke^{ik\theta}\leqq a_0\max_{0\leq k\leq n}[1+r^{k+1}]/|1-re^{i\theta}|.$$

This is proved very simply by writing the sum in the left member in the form $\sum b_kz^k$, where $b_k=a_k/r^k$ and $z=r\exp(i\theta)$, and then applying the Abel partial summation formula and using the fact that the b's are positive and monotone decreasing. In case the numbers a_n are complex, the same partial summation formula is applied directly to $\sum a_k\exp(ik\theta)$ to show that it is dominated by the product of

$$|a_0-a_1|+|a_1-a_2|+\cdots+|a_{n-1}-a_n|+|a_n|$$

and the greatest of the $n+1$ numbers $\sin\frac{1}{2}(k+1)\theta/\sin\frac{1}{2}\theta$, $k=0, 1, \cdots, n$. *R. P. Agnew* (Ithaca, N. Y.).

Sandham, H. F. Three summations due to Ramanujan. Quart. J. Math., Oxford Ser. (2) 1, 238–240 (1950).

The author presents simple proofs of three results due to Ramanujan [Collected Papers, p. 326]:

(1) $$\sum_1^\infty n^{11}/(e^{2\pi n}-1)=1/24;$$

(2) $$\sum_1^\infty n/(e^{2\pi n}-1)=1/24-1/8\pi;$$

(3) $$\sum_1^\infty \coth \pi n/n^7=19\pi^7/56700.$$

The proofs of (1) and (2) are based on the partial fraction expansion of $(e^{2\pi z}-1)^{-1}$, while (3) uses the expansion of $\coth \pi x$. The method will work for the more general sum $\sum_1^\infty n^{4p+1}(e^{2\pi n}-1)^{-1}$, where p is any integer. *J. Lehner.*

Young, F. H. A note on summation. Amer. Math. Monthly **57**, 625 (1950).

Palmer, Karl Otto. Eine Verallgemeinerung zweier Sätze von Mertens und Hardy auf Reihen negativer C-Summierbarkeitsordnung. Arch. Math. **2**, 258–266 (1950).

The author proves two theorems on summability of Cauchy product series by Cesàro methods C_{-k}, where k is a nonnegative integer. For Cesàro summability of negative integer order where the ordinary Cesàro formulas do not apply, the exposition of Lyra [Math. Z. **49**, 538–562 (1944); these Rev. **6**, 209] is followed. If $\sum a_n$ and $\sum b_n$ are summable C_{-k} to A and B, respectively, and if at least one of the series is absolutely summable C_{-k}, then the Cauchy product series $\sum c_n$ is summable C_{-k} to AB. If $\sum a_n$ and $\sum b_n$ are summable C_{-k} to A and B, respectively, and if both series are bounded C_{-k-1}, then the Cauchy product series $\sum c_n$ is summable C_{-k} to AB. The series $\sum \sin(\log n)/n \log n$ is given as an example of a conditionally convergent series which is summable C_{-k} for each $k=1, 2, \cdots$.

R. P. Agnew (Ithaca, N. Y.).

Andersen, A. F. On difference transformations. Mat. Tidsskr. B. **1950**, 110–122 (1950). (Danish)

Given an infinite sequence $\{a_\nu\}$ the differences of order α are defined by

$$\Delta^\alpha a_\nu = \sum_{\rho=0}^{\infty} A_\rho^{-\alpha-1} a_{\nu+\rho}, \qquad \nu=0, 1, \cdots, A_\rho^\alpha = \binom{\nu+\rho}{\nu}$$

if the series converge (all converge or none). A basic question in this theory is whether or not (*) $\Delta^\alpha(\Delta^\beta a_\nu)=\Delta^{\alpha+\beta}a_\nu$. The author [Thesis, University of Copenhagen, 1921] proved that, for a null sequence, (*) is valid for $\alpha\geqq -1$, $\beta\geqq 0$, $\alpha+\beta\geqq 0$. In this paper he shows that if (*) is valid for $\alpha=-r-t-1$, $\beta=r+1$ [misprint $r+t$ instead of $r+1$ in the statement of theorem VI, p. 118], where $r\geqq 0$, $t\geqq 0$, then it is also valid for $\alpha\geqq -r-t-1$, $\beta\geqq -t$, $\alpha+\beta\geqq -t$. For $r=0$, $t=0$ this reduces to the previously proved case, since $a_\nu=o(1)$ is equivalent to $a_\nu=\Delta^{-1}(\Delta^1 a_\nu)$. He also proves that if $\sum A_\nu{}^\gamma \Delta^{r+1}a_\nu$ converges, then (*) is valid for $\alpha\geqq -r-1$, $\beta\geqq 0$, $\alpha+\beta\geqq 0$, while the convergence of $\sum A_\nu{}^\gamma a_\nu$, $\gamma\geqq 0$, implies (*) if $\alpha, \beta, \alpha+\beta\geqq -\gamma-1$. The relations to investigations of Izumi [Tôhoku Math. J. **27**, 324–331 (1926)] and Iyengar [Proc. Indian Acad. Sci. Sect. A. **7**, 399–410 (1938); **8**, 20–38, 135–144 (1938)] are discussed.

E. Hille (New Haven, Conn.).

Bohr, Harald. On multiplication of summable Dirichlet series. Mat. Tidsskr. B. **1950**, 71–75 (1950).

If $\sum a_n n^{-s}$ is summable (C, r_1) and $\sum b_n n^{-s}$ is summable (C, r_2), $r_1\leqq r_2$, both for $\sigma>0$, then the product series $\sum c_n n^{-s}$, $c_n=\sum a_j b_k$ for $jk=n$, is summable (C, R), $0\leqq R\leqq r_1+r_2+1$ for $\sigma>\Phi$, where Φ is r_2-R or $\frac{1}{2}(r_1+r_2+1-R)$ according as $R+r_1+1$ is $\leqq r_2$ or $>r_2$. For every choice of r_1, r_2, R there exist Dirichlet series for which the abscissa of summability of order R is exactly Φ so the theorem is the best of its kind. The first part follows readily from the convexity properties of the summability function of the product series. In a recent paper [Math. Z. **52**, 709–722 (1950); these Rev. **12**, 170] the author gave an ingenious construction of three Dirichlet series f_1, f_2, f_3 with $f_3=f_1f_2$ such that the corresponding Lindelöf μ-functions satisfy $\mu_3=\mu_1+\mu_2$. This construction also shows that the expression found for Φ is the best possible.

E. Hille (New Haven, Conn.).

Gaier, Dieter. Über stetiges und asymptotisches Verhalten von Potenzreihen und Dirichletschen Reihen am Rande von Summationsgebieten. Math. Z. **53**, 291–308 (1950).

The author considers extensions of Abel's theorem in which more precise information concerning the behavior of a function defined by a power series is obtained at a boundary point of the domain of summability by imposing conditions on the rate at which the means approach their limit. In the simplest case, let $f(z)=\sum_0^\infty a_\nu z^\nu$ for $|z|<1$, let $C_k(n, s_\nu)$ be the nth Cesàro mean of order k of the sequence of partial sums at $z=1$, and suppose that

$$(1) \qquad C_k(n, s_\nu)=n^q[A+o(n^{-(1-\alpha)(1+k+q)})], \qquad n\to\infty,$$

where $k\geqq 0$, $1\geqq\alpha>0$, $q>-1$. Then

$$(2) \qquad f(z)\cong A\frac{\Gamma(1+k+q)}{\Gamma(1+k)}(1-z)^{-q},$$

where $z\to 1$ along a path on which ultimately $0\leqq r\leqq 1-c|\varphi|^{1/\alpha}$, $z=re^{i\varphi}$, c fixed positive but arbitrary. The conclusion is no longer necessarily valid if $z\to 1$ along a path which has a higher order of osculation with the unit circle at $z=1$ than that permitted by the theorem. If o is replaced by O in the assumption, the conclusion becomes $f(z)=O[(1-z)^{-q}]$ for the same type of approach. Various special cases were previously known; the author lists $q=0$, $k=0$, and $q=0$ with $\alpha=1$ or $\frac{1}{2}$. The theorem is extended to other types of summability, for instance, the Euler-Knopp E_p method and its generalizations, the S_α method of W. Meyer-König [Math. Z. **52**, 257–304 (1949); these Rev. **11**, 242], and a variant of Borel summability, in each case followed by C_k. In the C_kB-case the assumption that the C_kB limit of $\sum_0^\infty a_\nu z^\nu$ exists for real z, $0<z<1$, implies that $f(z)$ is holomorphic in $|z-\frac{1}{2}|<\frac{1}{2}$ and admits of a representation of the form $(1-z)^{k+1}z^{-k-2}$ times a Laplace integral in the variable $(1-z)/z$. If the $C_kB(\omega, s_\nu)$ satisfy (1) with ω instead of n, then $f(z)$ satisfies (2) for $z\to 1$ in the domain $0\leqq\rho\leqq\frac{1}{2}-c|\psi|^{1/\alpha}$, where $z=\frac{1}{2}+\rho e^{i\psi}$. Similar extensions are given for Dirichlet series with the aid of the means of M. Riesz. If $f(s)=\sum_1^\infty a_\nu e^{-\lambda_\nu s}$, if $R_k(\omega, s_\nu)$ satisfies (1) with ω instead of n, then $f(s)$ satisfies (2) with s instead of $1-z$ in the right member for $s=\sigma+it\to 0$ with $0<\sigma$, $0\leqq|t|\leqq C\sigma^\alpha$. The author states that the results can be extended to Laplace integrals.

E. Hille (New Haven, Conn.).

Rajagopal, C. T. On converse theorems of summability: addendum. Math. Gaz. **34**, 125 (1950).

This supplements work of the author [same Gaz. **30**, 272–276 (1946); these Rev. **8**, 375] on Tauberian theorems for Riesz $R(\lambda_n, 1)$ summability. If $\sum a_n$ is summable $R(\lambda_n, 1)$ to s and

$$\frac{1}{\lambda_n-\lambda_{n-1}}\log\frac{|a_n|(\lambda_{n-1}-\lambda_{n-2})}{|a_{n-1}|(\lambda_n-\lambda_{n-1})}\leqq -\frac{1}{\lambda_{n-1}},$$

then $\sum a_n$ must converge to s.

R. P. Agnew.

Rajagopal, C. T. A note on "positive" Tauberian theorems. J. London Math. Soc. **25**, 315–327 (1950).

Suppose that $\beta>-1$ and that, for $u\geqq 0$, $\psi(u)$ is positive with a continuous derivative and that $u^\beta\psi(u)$ is ultimately monotonic. Suppose also that $\int_0^\infty \psi(u)\,du=1$, $\int_0^\infty u^{ix}\psi(u)\,du\neq 0$ $(-\infty<x<\infty)$ and that $t\int_0^\infty \psi(ut)A(u)\,du\sim t^{-\beta}$ as $t\to+0$. It is proved that $\int_0^u A(x)\,dx\sim u^{\beta+1}/(\beta+1)$ as $u\to\infty$ provided that either (i) $A(u)\geqq 0$, or (ii) $\beta>0$ and $u^{-\beta}[A(u')-A(u)]$ is bounded below when $\lambda>1$ is fixed and $u<u'<\lambda u$. The stronger conclusion $A(u)\sim u^\beta$ is established when $uA(u)$

increases and it is shown that the condition $\psi(u)\geqq 0$ may be relaxed somewhat if $A(u)$ also increases. *H. R. Pitt.*

Delange, Hubert. Sur le théorème taubérien de Ikéhara. C. R. Acad. Sci. Paris **232**, 465–467 (1951).

Theorem: Let $G(s)=\int_0^{+\infty}e^{-st}\beta(t)\,dt$, where $\beta(t)$ is a real function defined for $t\geqq 0$, measurable and bounded over any finite interval, and where the integral is convergent for $\Re(s)>0$. Let there exist an integer $p\geqq 0$, and a real continuous and nondecreasing function $\gamma(t)$ such that (1) the integral $\int_0^{+\infty}e^{-tu}\gamma(u)\,du$ be convergent for $t>0$, and (2) $1/\beta(t)\sim t^{p+1}\int_0^{+\infty}e^{-tu}\gamma(u)\,du$ as $t\to\infty$. In case $\gamma(0)=0$, let further the integral $\int_0 \log[1/\gamma(t)]\,dt$ be convergent. This being the case, let (1) $\alpha(t)$ be real and nondecreasing for $t\geqq 0$; (2) the integral $\int_0^{+\infty}e^{-st}\alpha(t)\,dt$ converge for $\Re(s)>a$ $(a>0)$; (3) for each real y, there exist a positive function $\varphi_y(t)$ defined for sufficiently small and nonincreasing positive t such that the integral $\int_0\varphi_y(t)\,dt$ and $\int_0\varphi_y(t)\log[1/\gamma(t)]\,dt$ be convergent, and that when s tends to zero in the half-plane $\Re(s)>0$, $F^{(p)}(a+iy+s)=O(\varphi_y(r)/\gamma(r))$, where $r=|s|$ and $F(s)=\int_0^{+\infty}e^{-st}\alpha(t)\,dt-AG(s-a)$, $A>0$, $a>0$. Then we have $\alpha(t)\sim Ae^{at}\beta(t)$ as $t\to\infty$.

When we take $\gamma(u)=1$, $p=0$, $\beta(t)=1$, this theorem reduces to the Wiener-Ikehara theorem. The author states two corollaries of the above theorem susceptible of arithmetical applications. *S. Ikehara* (Cambridge, Mass.).

Gál, I. S. Sur les moyennes arithmétiques des suites de fonctions orthogonales. Ann. Inst. Fourier Grenoble **1** (1949), 53–59 (1950).

Rademacher [Math. Ann. **87**, 112–138 (1922), p. 122] has shown that if $\{\varphi_n(x)\}$ is a sequence of orthonormal functions in $a\leqq x\leqq b$, then $\sum_1^N\varphi_n(x)=o(N^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\epsilon})$ almost everywhere in (a, b). The author proves that

$$\sum_1^N(1-(n-1)/N)\varphi_n(x)=o(N^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\epsilon})$$

almost everywhere in (a, b). This result is deduced from a theorem due to the author and Koksma [C. R. Acad. Sci. Paris **227**, 1321–1323 (1948); the author, ibid. **228**, 636–638 (1949); these Rev. **10**, 292, 550]. No proof of this general theorem was given at the time, but the author now gives a proof of a special case of the theorem of the author and Koksma from which his theorem on orthogonal functions follows. *A. C. Offord* (London).

Fourier Series and Generalizations, Integral Transforms

Schmetterer, Leopold. Über einen Satz von Hardy und Littlewood. Monatsh. Math. **54**, 135–139 (1950).

The author proves that if $\varphi(t)$ is an even Cauchy-Lebesgue integrable function such that $\varphi(t)=O(t^{-\gamma})$ as $t\to 0$ $(\gamma>1)$ and if the nth arithmetic mean of the Cauchy-Fourier series of $\varphi(t)$ is of order $o(n^{-\alpha})$ $(0<\alpha<1, \alpha/(1-\alpha)>\gamma)$, then $\int_0^t\varphi(u)\,du=o(t)$ as $t\to 0$. This theorem is a Tauberian theorem of the same type as those due to Hardy and Littlewood [Ann. Scuola Norm. Super. Pisa (2) **3**, 43–62 (1934)], C. T. Loo [Trans. Amer. Math. Soc. **56**, 508–518 (1944); these Rev. **6**, 126], and the reviewer [see the following review]. *S. Izumi* (Tokyo).

Izumi, Shin-ichi. Notes on Fourier analysis. XXXV. Tôhoku Math. J. (2) **1**, 285–302 (1950).

Let $\varphi(t)=f(x+t)+f(x-t)-2f(x)$ and let x be a point at which $\int_0^t\varphi(u)u^{-\alpha-1}\,du$ exists in the Cauchy sense or where $\int_0^t\varphi(u)(t-u)^{\alpha-1}\,du=O(t^\alpha)$ as $t\to 0$. Known results concerning the order of the partial sums $S_n(t)$ of the Fourier series of $f(t)$ at $t=x$ are shown to be best possible. Part II of the paper is concerned with a comparison of the strength in the summing of Fourier series of the methods (K, j), $j=1, 2$, introduced by Zygmund [Studia Math. **10**, 97–103 (1948); these Rev. **10**, 31] with the Riemann methods R_j, $j=1, 2$. The author finds R_1 and $(K, 1)$ equivalent and the R_2 and $(K, 2)$ methods exclusive. The third portion of the paper is devoted to the establishment of instances in which order relations $\sigma_n^{(\beta)}=O(n^\gamma)$ as $n\to\infty$ for the Cesàro means of order β of the Fourier series of an even function $f(t)$ imply $\int_0^t f(u)(t-u)^{\alpha-1}\,du=O(t^\alpha)$ as $t\to 0$. *P. Civin.*

Izumi, Shin-ichi. Notes on Fourier analysis. XXVI. Some negative examples in the theory of Fourier series. Tôhoku Math. J. (2) **2**, 74–95 (1950).

A number of examples are given which establish that various conditions for convergence or (C, α)-summability of a Fourier series at a point $t=x$ are not necessary. An example of the type of theorem is the following: There is a function $f(t)\in L_p$, $1\leqq p<\infty$, such that

$$\int_0^t\{f(x+u)+f(x-u)-2f(x)\}\,du\neq o(t),$$

while the Fourier series of $f(t)$ converges to $f(x)$ at $t=x$. *P. Civin* (Eugene, Ore.).

Pleade see Errata 1 p. 1003.

Izumi, Shin-ichi, and Sunouchi, Gen-ichirô. Notes on Fourier analysis. XXXIX. Theorems concerning Cesàro summability. Tôhoku Math. J. (2) **1**, 313–326 (1950).

The authors establish some new sufficient conditions for the (C, α)-summability of the Fourier series of a function $f(t)$ at a point $t=x$ as well as giving new proofs of some known sufficiency conditions. Let

$$\varphi_\alpha(t)=\int_0^t(t-u)^{\alpha-1}\{f(x+u)+f(x-u)-2f(x)\}\,du.$$

Then $\varphi_\alpha(t)=o(t^\alpha\log t^{-1})$ implies summability (C, α) at $t=x$ for $\alpha>0$ and $\varphi_\beta(t)=O(t^\gamma)$ for $0<\beta<\gamma$ implies (C, α)-summability at $t=x$ for $\alpha>\beta/(\gamma-\beta+1)$. *P. Civin.*

Szász, Otto. Gibbs' phenomenon for Hausdorff means. Trans. Amer. Math. Soc. **69**, 440–456 (1950).

The Hausdorff means of a sequence $\{S_n\}$, which include as special cases the means of Euler, Cesàro, and Hölder, are defined by $h_n=\sum_{j=0}^n\binom{n}{j}S_j\int_0^1 r^j(1-r)^{n-j}\,d\psi(r)$, $n=0, 1, 2, \cdots$, where $\psi(r)$ is of bounded variation in $0\leqq r\leqq 1$ and $\psi(0)=\psi(0+)=0$ and $\psi(1)=1$. For the Fourier series $\sum_{j=1}^\infty(1/j)\sin jt$ with partial sums $S_j=S_j(t)$ it is demonstrated that $h_n(t_n)\to\int_0^1\int_0^T(\sin ry/y)\,dy\,d\psi(r)$ as $nt_n\to T\leqq\infty$. The above result is applied to Hölder summability of order p. The author demonstrates the existence of a constant α, $0.58<\alpha<0.60$, such that the Gibbs phenomenon persists for $p<\alpha$ but vanishes for $p\geqq\alpha$. The value of α is larger than the corresponding constant $j=0.4395\ldots$ [Gronwall, Ann. of Math. (2) **31**, 233–240 (1930)] for the (C, p) methods equivalent to Hölder summability of order p. *P. Civin.*

Zamansky, Marc. Sur la sommation des séries de Fourier dérivées. C. R. Acad. Sci. Paris **231**, 1118–1120 (1950).

The author states without proof a few results concerning the summation of the Fourier series of a function f,

$$(*)\qquad f\sim\tfrac{1}{2}a_0+\sum_1^\infty(a_n\cos kx+b_n\sin kx)=\sum_0^\infty A_k(x),$$

differentiated termwise p times. For a given x, let

$$\Delta_1(t)=f(x+t)-f(x-t), \quad \Delta_2(t)=f(x+t)+f(x-t)-2f(x),$$
$$\Delta_k(t)=\Delta_{k-2}(t)-2^{k-2}\Delta_{k-2}(\tfrac{1}{2}t),$$

$k=3, 4, \cdots$. The generalized derivative D_p of order p of f at the point x is defined as the limit of $\Delta_p(t)/\lambda_p t^p$ for $t\to 0$, where λ_p is a normalizing constant depending on p only. The author is interested in defining a method of summation, of the form $\sum_{k=0}^{\infty} g(k/n)A_k(x)=T_n(x, f, g)$ and such that at every point x at which the generalized derivative $D_p f(x)$ exists not only do the polynomials T_n tend to f, but also $T_n^{(p)}\to D_p f(x)$. Corresponding results are also stated for the series conjugate to (*). *A. Zygmund* (Chicago, Ill.).

Cheng, Min-Teh. The Gibbs phenomenon and Bochner's summation method. II. Duke Math. J. **17**, 477–490 (1950).

[For part I see same vol., 83–90 (1950); these Rev. **11**, 659.] The author defines the Gibbs phenomenon for a sequence of functions of two variables, and proves that the circular Riesz mean of positive order of the Fourier series of $f(x, y)=f(x)\cdot g(y)$, $f(x)$ and $g(y)$ being of bounded variation, presents Gibbs' phenomenon at its points of discontinuity. *S. Izumi* (Tokyo).

Campbell, Robert. Sur les fonctions développables en séries de fonctions de Weber. C. R. Acad. Sci. Paris **231**, 1024–1026 (1950).

In essence, this note contains a somewhat weaker form of the equiconvergence theorem for Hermite series [Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, theorem 9.1.6; these Rev. **1**, 14]. *A. Erdélyi* (Pasadena, Calif.).

Udagawa, Masatomo. On lacunary non-harmonic trigonometric series. Kōdai Math. Sem. Rep. **1950**, 17–20 (1950).

The author gives extensions of several results concerning harmonic gap series to the nonharmonic case. As an example we quote the following result: Let $\lambda_{k+1}/\lambda_k>\lambda>1$ and $\sum_1^{\infty}|c_k|^2<\infty$. It is known [Kac, Duke Math. J. **8**, 541–545 (1941); these Rev. **3**, 107] that the series $\sum_1^{\infty} c_k e^{\alpha\lambda_k x}$ converges almost everywhere in $(-\infty, \infty)$. Denoting the sum of this series by $f(x)$ and setting $S_n(x)=\sum_1^n c_k e^{\alpha\lambda_k x}$, $S^*(x)=\sup_n|S_n(x)|$, the author shows that

$$\int_a^b |S^*(x)|^r\,dx\leqq D\int_a^b |f(x)|^r\,dx, \qquad r>1,$$

where D depends on a, b, r, and λ. The integral on the right hand side of the last inequality is finite. *M. Kac.*

Kawata, Tatsuo. On gap series. Kōdai Math. Sem. Rep. **1950**, 21–30, 40 (1950).

This paper is devoted to the study of various properties of gap series $\sum_1^{\infty} c_k\varphi(\lambda_k x)$, where $\varphi(x+2\pi)=\varphi(x)$, $\int_0^{2\pi}\varphi(x)\,dx=0$, and $\varphi(x)$ satisfies a Lipschitz condition of order α. The results are mostly extensions of known theorems to the case where the λ's are not necessarily integers. *M. Kac.*

Maruyama, Gisiro. On an asymptotic property of a gap sequence. Kōdai Math. Sem. Rep. **1950**, 31–32 (1950).

The author gives a simple proof of the following theorem: If $f(t+1)=f(t)$, $\int_0^1 f(t)\,dt=0$, and $f(t)$ satisfies a Lipschitz condition of order α, then

$$\limsup_{n\to\infty}\sum_1^n f(2^k t)/(2n\log\log n)^{\frac{1}{2}}=\sigma,$$

where $\sigma^2=\lim_{n\to\infty} n^{-1}\int_0^1(\sum_1^n f(2^k t)^2\,dt$. This theorem was first stated by Fortet [Studia Math. **9**, 54–70 (1940); these Rev. **3**, 169] who sketched a proof based on Doeblin's generalization of the law of the iterated logarithm to Markoff chains. The author deduces the theorem from Kolmogoroff's law of the iterated logarithm by applying an approximation procedure introduced by the reviewer [Ann. of Math. (2) **47**, 33–49 (1946); these Rev. **7**, 436]. *M. Kac.*

Hartman, S. Remarque au travail "Sur les bases statistiques." Studia Math. **11**, 197–199 (1950).

Simplification of the proof of a theorem in the paper cited in the title [Studia Math. **10**, 120–139 (1948); these Rev. **10**, 603]. *J. L. Doob* (Urbana, Ill.).

Ugaheri, Tadashi. On the abscissa of convergence of Laplace-Stieltjes integral. Ann. Inst. Statist. Math., Tokyo **2**, 1–3 (1950).

The author shows that the abscissa of convergence σ_c of the Laplace transform $\int_0^{\infty} e^{-st}\,d\alpha(t)$ is given by the formula $\sigma_c=\limsup t^{-1}\log|\int_{[t]}^{t} d\alpha(t)|$, where $[t]$ is an integer such that $t-1\leqq[t]<t$. The corresponding classical formulas distinguish between the cases $\sigma_c\geqq 0$ and $\sigma_c<0$.
I. I. Hirschman, Jr. (St. Louis, Mo.).

San Juan, Ricardo. Caractérisation de la transformation de Laplace par la loi de composition appelée règle de la "Faltung." Portugaliae Math. **9**, 177–184 (1950).

The author shows that the Laplace transform

$$\mathfrak{L}\varphi=\int_0^{\infty}\varphi(t)e^{-tz}\,dt$$

is essentially characterized by the equation $\mathfrak{L}\varphi\cdot\mathfrak{L}\varphi_1=\mathfrak{L}\varphi_1*\varphi$, where $\varphi_1*\varphi=\int_0^t\varphi_1(u)\varphi(t-u)\,du$. *I. I. Hirschman, Jr.*

Holzer, L. Zur Laplace-Transformation der Besselschen Funktionen. Revista Mat. Hisp.-Amer. (4) **10**, 16–29, 51–69 (1950).

Let $Lf(t)=\int_0^{\infty} e^{-zt}f(t)\,dt$ be the Laplace transform of $f(t)$, let k be any (fixed) nonnegative integer, μ a (fixed) real number, and put $P(z)=(k+1)z^{k+2}-(k+2)z^{k+1}p-1$. Let $z=p_1$ be that root of $P(z)=0$ which has largest modulus, and let C be a simple closed curve around p_1 such that no other root of $zP(z)=0$ lies on or inside C. The author defines $\Xi_{\mu k}(t)$ as the function whose Laplace transform is $I_{\mu k}=(1/\pi i)\int_C(z^{-\mu}/P(z))\,dz$. The integral $I_{\mu k}$ is in fact a Laplace transform if $\mu+k>-1$. For $k=0$ one obtains a Bessel function; for positive integer k a generalised hypergeometric series ${}_0F_{k+1}$. The author investigates several properties of his functions which are generalizations of certain properties of Bessel functions. [The paper contains many misprints; in particular, the factor p in the second term of $P(z)$ seems to have been omitted.] *A. Erdélyi.*

***Meijer, C. S. Berekening van bepaalde integralen met behulp van de omkeerstelling van Mellin en de integralen van Barnes.** [**Evaluation of definite integrals by means of Mellin's inversion theorem and Barnes' integrals**]. Lectures at the Mathematisch Instituut, Groningen, 1950. 21 pp.

In the last fifteen years or so the author investigated a large number of integral representations and integral for-

mulas for functions of the hypergeometric type. One of the principal tools in these investigations is Mellin's inversion theorem in combination with Barnes' integrals. The pamphlet under review contains a general exposition of this method, with illustrative examples involving Bessel functions and other functions of the hypergeometric type: it is a summary of lectures given first, in 1949, in the Amsterdam Mathematical Centre and repeated, in 1950, in an expanded form, in Groningen. *A. Erdélyi* (Pasadena, Calif.).

*van der Pol, Balth., and Bremmer, H. **Operational Calculus, Based on the Two-Sided Laplace Integral.** Cambridge, at the University Press, 1950. xiii+415 pp. $10.00.

This book is intended as a treatise on the application of the operational calculus in its modern form to mathematics, physics, and engineering. The authors have adopted a rather attractive and unique mode of presentation to cater to the needs of applied mathematicians and engineers. They seem to feel that on the one hand modern applications of the operational calculus require theorems under rather general conditions, and on the other hand the readers they have in mind are not interested in, or not able to follow, proofs under such general conditions. They freely use concepts like functions of bounded variation, upper and lower limits, Cesàro summability. These concepts are carefully explained, and a large number of illustrative examples are used to clarify the precise statement, the meaning of the various conditions and the role they play, and also to teach the practical use of the theorem. Proofs of the basic theorems are omitted as a rule, but references are given to standard text-books (mostly the books by Doetsch or Widder). The great advantages of this mode of presentation, when it is handled as skillfully as in the book under review, are fairly obvious; so are its limitations. Only the fundamental theorems can be discussed on such a lavish scale as described above, and with many other theorems the conditions of validity will not always be clear to the nonmathematical reader (although clearly recognizable for the professional mathematician). As an example we may mention the differentiation rule in IV. 8. The formal derivation is accompanied by the remark that the validity of the rule depends on the interchangeability of two limiting processes, examples are given to illustrate different possibilities, but no precise conditions of validity are given.

Another valuable feature of the book is the large number, and wide range, of examples. Besides the more obvious kind, from electrical network theory, etc., there is an unusually large and good selection of examples showing the technique of using operational calculus for the investigation of special functions (Bessel and Legendre functions, hypergeometric functions, etc.), and as a novelty in a book written primarily for engineers, there are a large number of examples concerning the functions of number theory. Yet another novel feature is the inclusion of a chapter on operational calculus in several variables.

The operational calculus is based on the two-sided Laplace transform (1) $f(p)=p\int_{-\infty}^{\infty}e^{-pt}h(t)\,dt$ rather than on the one-sided transform (with 0 and ∞ as the limits of integration). The authors regard this as the most novel and most important feature of their book, and hold that it enables them to discuss a larger class of functions, simplifies the so-called rules, and leads to a more rigorous treatment. This claim was put forward in two notes [Nederl. Akad. Wetensch., Proc. 51, 1005–1012, 1125–1136 = Indagationes Math. 10, 338–345, 349–360 (1948); these Rev. 10, 294] but never quite substantiated. It seems to the reviewer that every complete discussion should (and the present treatise, like several others, does) include both the one-sided and the two-sided transformation, and apart from personal taste there is little reason to choose one rather than the other as a starting point. In (1), $h(t)$ is called the original, and $f(p)$ the image. One-sided Laplace transforms are introduced by considering originals of the form $U(t)h(t)$, where $U(t)$ is Heaviside's unit function ($=0$ for $t<0$ and $=1$ for $t>0$). A careful discussion of the unit function and of its derivatives, and the interpretation of the image of $U'(t)$ as a Stieltjes integral provide a satisfactory basis for the use of these functions in operational calculus.

After two introductory chapters, the mathematical theory is presented in five chapters. Notable features are a thorough investigation of the region of convergence, both of (1) and of the inversion integral, and an unusually complete discussion (for a book of this kind) of asymptotic relations. More than one half of the book is devoted to applications of the operational calculus to differential, difference, and integral equations, and to some other problems. In these applications the emphasis is mostly on the technique. No general theorems are formulated, and in the case of partial differential equations, where it is known that the tentative solutions must be justified, no such verification is carried out. A brief summary of the contents follows.

Chapter I. General introduction. Chapter II. The Fourier integral as basis of the operational calculus. Chapter III. Computation of images of elementary, and of some higher transcendental, functions. Chapter IV. The "rules" (similarity rule, shift rule, composition product, differentiation and integration rule, and the like). Chapter V. The unit function and its derivatives, in particular the delta or impulse function. Chapter VI. A thorough discussion of the convergence of the Laplace integral, behaviour on the boundary of the domain of convergence, the complex inversion formula, uniqueness. Chapter VII. Asymptotic relations (both of Abelian and Tauberian character), manipulation of infinite series of certain types, the Post-Widder inversion formula. Chapter VIII. Linear differential equations with constant coefficients. Chapter IX. Systems of such equations. Chapter X. Linear differential equations with variable coefficients, in particular, the various differential equations associated with hypergeometric and confluent hypergeometric functions. Chapter XI. More advanced rules such as the images of $h(t^{-1})$, $h(t^2)$, $h(e^t)$, the originals belonging to $p^k f(p^{-1})$, and the like, and some integral identities. Chapter XII. Step functions, the saw-tooth function, and arithmetic functions. Chapter XIII. Difference equations. Chapter XIV. Integral equations. Chapter XV. Partial differential equations in the operational calculus in one variable. Chapter XVI. Operational calculus in several variables and its application to partial differential equations. Chapter XVII. "Grammar," being a collection of general formulas and rules of operational calculus. Chapter XVIII. "Dictionary," 27 pages of classified transform pairs.

The book is very well written, and may be read with profit by both mathematicians and engineers. The mathematical treatment is more advanced than in most other books devoted primarily to applications, and on this account the work is not suitable as a text-book except for a graduate course for engineers or physicists of unusually strong mathematical background, and then in connection with selected parts of a mathematical text (such as Doetsch, Titchmarsh,

or Widder) giving adequate proofs of the basic theorems. The translation (edited by C. J. Bouwkamp) preserves the flavour of the Dutch original, and the printing is excellent. *A. Erdélyi* (Pasadena, Calif.).

***McLachlan, N. W., et Humbert, Pierre. Formulaire pour le calcul symbolique.** 2d ed. Mémor. Sci. Math., no. 100. Gauthier-Villars, Paris, 1950. 67 pp. 350 francs.

This appears to be largely a photographic reproduction of the first edition [1941; these Rev. 3, 243]. Misprints and errors have been corrected in the present edition, but some errors still remain, and in at least one instance (p. 14, second formula) the removal of an error resulted in a misprint. The pamphlet reviewed below contains a list of corrections (to the first edition) which is more complete than that used in preparing the present second edition; but even that list is not entirely complete, and it contains a misprint in the item referring to the last formula on p. 48 of the first edition. Errors, removed and still remaining, are mentioned in this review for the information of prospective users, not in a spirit of adverse criticism. Experience shows that it is impossible to achieve a book of this nature completely free of errors. Actually, there are comparatively fewer errors in this book than in many other similar compilations, a circumstance that contributes greatly to the usefulness of the work. For additional material see the following review. *A. Erdélyi* (Pasadena, Calif.).

***McLachlan, N. W., Humbert, P., et Poli, L. Supplément au formulaire pour le calcul symbolique.** Mémor. Sci. Math. no. 113. Gauthier-Villars, Paris, 1950. 62 pp. 450 francs.

The present collection of formulas is a supplement to the well-known work by McLachlan and Humbert [cf. the preceding review and reference cited there] and is planned for use in conjunction with it. Apart from a list of corrections to the first edition of the book reviewed above (many of these corrections have already been carried out in the second edition), the organization of the supplement closely parallels that of the original work and need not be described here. It will be sufficient to mention some of the more important among the approximately 400 new formulas.

The list of definitions of functions has been enlarged to include several functions not previously listed, and also to provide more specific notations in some cases. For instance, there is now a special notation for the error function with imaginary variable; Ei (x) is now reserved for the exponential integral defined in the complex plane cut along the positive real axis, and the principal value of the exponential integral for positive real x is denoted by Ei* (x), thereby eliminating confusion. The operational image of $f(t)$, that is, $p\int_0^\infty e^{-pt}f(t)\,dt$, is denoted by $\varphi(p)$, and the relationship is indicated as $f(t)\supset\varphi(p)$. In part B, general formulas, there are now extensive sections on so-called transformations, of the first group if they relate images of $f(t)$ and of $g(t)f(h(t))$, of the second group if they relate originals belonging to $\varphi(p)$ and to $\psi(p)\varphi(\lambda(p))$, and of the third group if they relate $f(t)$ to the operational image of $g(t)\varphi(h(t))$. A section is devoted to so-called sequences which are essentially chains of transformations. There is also a collection of formulas useful in connection with the operational treatment of differential equations with variable coefficients, and other additions. Part C contains additions to almost every section of the corresponding part of the "Formulaire." Among the sections which have received especially important, or especially numerous, additions are: exponential integrals and related functions; error functions; discontinuous functions. New sections are devoted to: trigonometric functions of higher order; various functions related to the gamma function; the functions $\nu(t)$, $\mu(t, m)$, and related functions; and Mathieu functions. There is an addition to the bibliography. It is regrettable that Doetsch's Tabellen zur Laplace-Transformation . . . [Springer, Berlin-Göttingen, 1947; these Rev. 9, 237] has not been mentioned, and the omission of McLachlan's Modern Operational Calculus . . . [Macmillan, London, 1948; these Rev. 9, 581] is odd.

Every such collection must prove itself in actual use. The first impression is decidedly favourable. Inconsistencies, such as listing $|\cos t|$ under trigonometric functions, while $|\sin t|$ is listed under discontinuous functions in the "Formulaire," can hardly be avoided, and will no doubt be removed in a later edition. The reviewer has not carried out any extensive checking operations and can offer no opinion on the accuracy of the work. The misprints that have been noted accidentally are very few. *A. Erdélyi.*

Mikusiński, Jan G.-. Sur le calcul opératoire. Časopis Pěst. Mat. Fys. **74** (1949), 89–94 (1950). (French. Polish summary)

Summary of a lecture.

Berge, Claude. Sur un nouveau calcul symbolique et ses applications. J. Math. Pures Appl. (9) **29**, 245–274 (1950).

The first image of a function $\varphi(t)$ on the nonnegative integers to complex numbers is $f(x)=\sum_{n=0}^{\infty}x^n\varphi(n)/n!$, and the second image of $\varphi(t)$ is $\mathfrak{F}(p)=p\int_0^\infty e^{-px}f(x)\,dx$. Thus the symbolic calculus discussed in this paper is a combination of generating functions and Laplace transforms. The calculus is applied to combinatorial analysis, Bernoulli numbers, summability factors, difference and differential equations. *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Ostrowski, A. M. Note on Vincent's theorem. Ann. of Math. (2) **52**, 702–707 (1950).

The following theorem was proved by Vincent in 1834: Let $f(x)=A_0x^n+\cdots+A_n$, $A_0\neq0$, a real polynomial with the n distinct zeros $x_1, x_2, \cdots, x_n$, be transformed by the successive substitutions $x=X_1$, $X_k=a_k+(1/X_{k+1})$, $a_k\geqq1$, $k=1, 2, \cdots, m$, into the polynomial $f_m(X_m)=A_0^{(m)}X_m^n+\cdots+A_n^{(m)}$. Then, for all $m>M$, there is at most one variation of signs among the coefficients $A_k^{(m)}$ of $f_m(X_m)$. Vincent, however, did not furnish a value for M. One estimate for M was given by Uspensky [Theory of Equations, McGraw-Hill, New York, 1948, pp. 298–303]. A better estimate, developed in the present note, is the following: Let $D=\min|x_j-x_k|$, $j\neq k$, and let $a_1, a_2, \cdots$ be an arbitrary sequence of positive numbers. Let

$$\frac{P_m}{Q_m}=a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_m}}},\qquad Q_m>0,$$

be the mth convergent of this continued fraction. Then, if $Q_{m-1}Q_mD\geqq\sqrt{3}$ for some $m\geqq2$, the polynomial

$$F(X)=(Q_mX+Q_{m-1})^nf[(P_mX+P_{m-1})/(Q_mX+Q_{m-1})]$$

has at most one variation of signs. To prove this theorem, the author establishes conditions, some necessary and sufficient, for certain polynomials $f(x)$ with positive coefficients to have the property that the product $[(x-\alpha)f(x)]$ has one variation of signs for any positive α. *M. Marden.*

Vicente Gonçalves, J. Calcul abrégé d'une suite de Sturm. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1**, 189–200 (1950).

This is an expository article dealing with the calculation of Sturm's series and its application to determine the number of roots of a polynomial having real or imaginary parts of one sign. *H. S. Wall* (Austin, Tex.).

Görk, Eugen. Stabilitätskriterien. Arch. Elektr. Übertragung **4**, 89–96 (1950).

The author discusses several stability criteria related to those of Hurwitz and Nyquist. He applies them to some simple algebraic and transcendental transfer functions. *P. Franklin* (Cambridge, Mass.).

Ivanović, B. Sur la tendance de distribution des zéros d'un polynôme donné. Bull. Soc. Math. Phys. Serbie **2**, nos. 1–2, 49–54 (1950). (Serbo-Croatian. French summary)

The center of gravity and the moment of inertia of the n zeros of a polynomial can be expressed in terms of a_0, a_1, a_2. The author calculates the regression lines as if this had a statistical meaning. *W. Feller* (Princeton, N. J.).

Verblunsky, S. A theorem on cubic polynomials. Arch. Math. **2**, 281–282 (1950).

The author proves that, a necessary and sufficient condition that $y^3+\alpha y^2+\beta y+1>0$ whenever $y\geqq 0$, is that $\alpha\beta+6(\alpha+\beta)+9+2(\alpha+\beta+3)^{\frac{1}{2}}>0$. It is noted that any cubic polynomial can be reduced to the above form by a suitable transformation. *A. W. Goodman* (Lexington, Ky.).

Olds, C. D. The best polynomial approximation of functions. Amer. Math. Monthly **57**, 617–621 (1950).

Expository paper.

Rau, Heinz. Über die Entwicklung einer stetigen Funktion nach Jacobischen Polynomen. Arch. Math. **2**, 251–257 (1950).

Following the method of Prüfer used in the case of Sturm-Liouville series [Math. Ann. **95**, 499–519 (1926)], the author discusses the expansion of a function $f(x)$ in a series of Jacobi polynomials, $-1\leqq x\leqq 1$. Assuming that $f(x)$ is continuous and has a piece-wise continuous derivative, the series is proved to be uniformly convergent in each interval $[-1+\epsilon, 1-\epsilon]$. The uniqueness of the expansion is also discussed. *G. Szegö* (Stanford University, Calif.).

Rymarenko, B. A. More about polynomials which are monotonic on the whole real axis. Doklady Akad. Nauk SSSR (N.S.) **75**, 5–6 (1950). (Russian)

The author continues his work [same Doklady (N.S.) **71**, 1029–1032 (1950); these Rev. **11**, 662] on the smallest variation L_{2m+1} on $(-1, 1)$ of a polynomial $y(x)$ of degree $2m+1$ which increases on the whole real axis and is subjected to additional conditions. Here the conditions are $y'(x_i)=s_i^2\geqq 0$ $(i=1, \cdots, p\leqq m)$; an explicit solution is given when x_i are roots of the Legendre polynomial of degree m, and the asymptotic behavior of L_{2m+1} is found when x_i are arbitrary in $(-1, 1)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Ferrer Figueras, Lorenzo. On a possible extension of some properties of Legendre polynomials. Revista Mat. Hisp.-Amer. (4) **10**, 12–15 (1950). (Spanish)

The coefficient $P^q_{n,p}(x)$ of z^n in the expansion of $[1-x^q+(x-z)^q]^{-1/p}$ in ascending powers of z is a polynomial in x. The author states that these polynomials satisfy a linear differential equation. He obtains the differential equation and proves the orthogonal property only in the particular case $q=2$ (when the $P^q_{n,p}$ are Gegenbauer polynomials). *A. Erdélyi* (Pasadena, Calif.).

Seidel, W., and Szász, Otto. On positive harmonic functions and ultraspherical polynomials. J. London Math. Soc. **26**, 36–41 (1951).

Following the usual notation of the ultraspherical polynomials $P_n^{(\lambda)}(x)$, the authors prove that

$$\tfrac{1}{2}+\sum_{n=1}^{\infty}(P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1))z^n$$

has a positive real part in the unit circle $|z|<1$ provided $x=\cos\theta$, $0<\theta<\pi$, and $\lambda>0$. The proof is based on an integral formula due to Gegenbauer which is a generalization of the Laplace integral formula of the Legendre polynomials. The assertion is equivalent to the fact that the Hermitian forms $\sum_{\mu,\nu=0}^{n}[P_{\mu-\nu}^{(\lambda)}(x)/P_{\mu-\nu}^{(\lambda)}(1)]u_\mu\bar{u}_\nu$ are positive definite. According to a remark of the reviewer the assertion does not hold for $-\frac{1}{2}<\lambda<0$. *G. Szegö.*

Steffensen, J. F. On a special type of polynomials. Mat. Tidsskr. B. **1950**, 6–9 (1950).

In an earlier paper [Acta Math. **73**, 333–366 (1941); these Rev. **3**, 236] the author defined an operator θ. In the paper under review this operator is used to set up a sequence of polynomials $g_n(x)$ of degree n, and more generally a set of polynomials $g_n(x, y)$ in two variables. The properties of these polynomials are briefly mentioned. *W. E. Milne.*

Special Functions

***Ahiezer, N. I. Èlementy teorii èlliptičeskih funkcii.** [**Elements of the Theory of Elliptic Functions**]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948. 291 pp.

This book is intended for readers with relatively little mathematical preparation. The arrangement of material is indicated by the following abbreviated chapter headings: general theory, modular functions, Weierstrassian functions, theta functions, Jacobian functions, transformation of elliptic functions, elliptic integrals, conformal mappings, some extremal problems (best approximation by rational functions to a function which takes the values $+1$ on one interval and -1 on another, and equivalent extremal problems), miscellaneous applications (e.g., the Dirichlet problem for a ring, elliptic coordinates and Lamé's equation, Picard's theorem), table of formulas (20 pages). *R. P. Boas, Jr.*

Buchner, P. Bemerkungen zur Stirlingschen Formel. Elemente der Math. **6**, 8–11 (1951).

Bose, N. N. Some properties of MacRobert's E-function. Bull. Calcutta Math. Soc. **42**, 94–98 (1950).

Recurrence relations and integral formulas for a particular case of MacRobert's E-function. The particular E-function involved can be expressed (on the author's showing) in terms of Whittaker's W-function, and most of the formulas obtained are paraphrases of known formulas for Whittaker functions.

A. Erdélyi (Pasadena, Calif.).

Srivastava, H. M. On Bateman's function and an allied function. Bull. Calcutta Math. Soc. **42**, 82–88 (1950).

Of the two functions defined by

$$K_n(x)\pm iT_n(x)=2\pi^{-1}\int_0^{1\pi}\exp\left[\pm i(x\tan\theta-n\theta)\right]d\theta$$

the first is Bateman's function, the properties of which are briefly recalled in this paper. Corresponding properties of $T_n(x)$ are established; in particular, it is shown that the difference and the differential equation satisfied by T_n differ only by the presence of a (constant) right hand side from the corresponding equations associated with K_n. In the second half of the paper the author derives a number of functional relations involving both K_n and T_n.

A. Erdélyi (Pasadena, Calif.).

Tsuchikura, Tamotsu. On the functions Ci (x, y) and Si (x, y). Tôhoku Math. J. (2) **2**, 68–73 (1950).

Let U_n be Lommel's function of two variables [Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press and Macmillan, New York, 1944, § 16.5; these Rev. **6**, 64]. Then $\mathrm{Ci}\,(x, y)=\int_x^\infty t^{-1}U_0(2t, 2y)\,dt$, $\mathrm{Si}\,(x, y)=\int_0^x t^{-1}U_1(2t, 2y)\,dt$. For these functions the author gives many formulas, among them, series expansions, integrals, and special values (for $y=0$ or $y=x$).

A. Erdélyi (Pasadena, Calif.).

Chaundy, T. W. Some hypergeometric identities. J. London Math. Soc. **26**, 42–44 (1951).

This note contains the identity

$$F(a, b; c; x)=h\sum_{n=0}^{\infty}\frac{(h-\alpha n+1)_{n-1}(e)_n}{n!(c)_n}$$
$$\times{}_4F_3\begin{bmatrix}a, b, 1+h/\beta, -n\\ e, h/\beta, h-\alpha n+1\end{bmatrix}(-x)^n{}_2F_1(e+n, h+\beta n; c+n; x),$$

$(\alpha+\beta=1)$, several particular cases, and some relations between generalized hypergeometric series of unit argument.

A. Erdélyi (Pasadena, Calif.).

Kopineck, Hermann-Josef. Austausch- und andere Zweizentrenintegrale mit $2s$- und $2p$-Funktionen. Z. Naturforschung **5a**, 420–431 (1950).

In the quantum mechanical theory of chemical bonds one encounters certain integrals involving (approximate) wave functions. In the paper under review the wave functions are exponential functions of the radial distance multiplied by polynomials either in the radial distance or in the Cartesian coordinates. Upon introduction of bipolar coordinates, many of the integrals can be evaluated in terms of elementary functions, the others can be expressed in terms of elementary functions and the incomplete gamma function. The paper contains more than five pages of formulas, and two pages of numerical values, of these integrals. *A. Erdélyi.*

Harmonic Functions, Potential Theory

Kjellberg, Bo. On the growth of minimal positive harmonic functions in a plane region. Ark. Mat. **1**, 347–351 (1950).

The author considers minimal positive harmonic functions in an unbounded plane region D. By the order of such a function u is meant $\limsup_{r\to\infty}\log\max u(re^{i\theta})/\log r$. It is shown that if $v_1, \cdots, v_n$ $(n\geqq 2)$ are positive minimal harmonic functions in D which vanish continuously at every finite boundary point and are such that no two are proportional, then $\sum_1^n\rho_k^{-1}\leqq 2$, where ρ_k is the order of v_k. The proof uses the argument of the Denjoy-Carleman-Ahlfors theorem and a preliminary theorem which states that a positive harmonic function dominated by a finite sum of positive minimal harmonic functions is a linear combination with nonnegative coefficients of the latter functions. Several consequences are treated. *M. Heins* (Providence, R. I.).

Verblunsky, S. Sur les fonctions préharmoniques. II. Bull. Sci. Math. (2) **74**, 153–160 (1950).

[Pour la première partie voir le même Bull. (2) **73**, 148–152 (1949); ces Rev. **11**, 357.] Démonstration du théorème: Soit f une fonction préharmonique sur le réseau des points entiers (x, y); si

$$(1)\qquad \liminf_{|x|+|y|\to\infty}\frac{f(x, y)}{|x|+|y|}\geqq 0,$$

alors f est une constante. En remplaçant l'hypothèse (1) par l'hypothèse plus forte que $f(x, y)\geqq 0$ partout, on retrouve un théorème de J. Capoulade [Mathematica, Cluj **6**, 146–151 (1932)]. *H. Cartan* (Paris).

Kunugui, Kinijiro. Étude sur la théorie du potentiel généralisé. Osaka Math. J. **2**, 63–103 (1950).

Cet article est consacré à la théorie du potentiel par rapport à un noyau $K(x)$ défini dans R^m et qui est une fonction >0 de la seule distance $|x|=r$ continue pour tout r, finie et décroissante pour tout $r>0$, nulle à l'infini, sousharmonique pour $x\neq 0$, enfin sommable sur tout compact. Frostman en avait jeté les bases dans sa thèse [Lund, 1935]; d'ailleurs les méthodes employées sont celles de cet auteur, mais leur application exige quelques préliminaires, parmi lesquels il faut signaler en premier lieu le théorème fondamental sur le signe de l'intégrale d'énergie (si les masses μ sont situées à distance finie et si $\iint K(x-y)\,d\mu(x)\,d\mu(y)$ a un sens, cette quantité est $\geqq 0$, nulle seulement si $\mu=0$).

Après quelques préliminaires relatifs à la notion de capacité [selon de la Vallée Poussin, Les nouvelles méthodes de la théorie du potentiel . . . , Actualités Sci. Ind., no. 578, Hermann, Paris, 1937; il s'agit seulement de la capacité intérieure] et au principe du maximum de Maria [Proc. Nat. Acad. Sci. U. S. A. **20**, 485–489 (1934)] et Frostman [loc. cit.], l'auteur montre notamment le théorème suivant (avec $m=3$ pour le langage, mais sous une forme un peu plus générale): Soient E un compact de capacité >0 et $f(x)$ une fonction sousharmonique $\leqq 0$ dans R^3, dont la restriction à E est continue; il existe alors une distribution positive et une seule portée par E, de masse totale donnée, et dont le potentiel U satisfait à $U(x)+f(x)=c$ (constante) sur E, sauf sur un sous-ensemble de capacité nulle, sur lequel on a $U(x)+f(x)<c$. Ce résultat est appelé "théorème du balayage" parce qu'on en déduit le balayage, au sens usuel, dans le cas newtonien (mais dans ce cas seulement, de sorte que la dénomination est discutable). Pour $f\equiv 0$, c'est le théorème d'équilibre. La démonstration de l'unicité repose

sur le théorème relatif au signe de l'intégrale d'energie, qui est établi à l'aide de la transformation de Fourier. Quelques remarques sur la condition de Poincaré généralisée terminent cet article dans lequel on trouvera également de nombreux renseignements sur les récents travaux effectués par les mathématiciens japonais sur des sujets voisins.

J. Deny (Strasbourg).

Ninomiya, Nobuyuki. Equilibrium potentials and energy integrals. Proc. Japan Acad. **26**, no. 1, 1–16 (1950).

Démonstration directe (sans usage de la transformation de Fourier) du théorème de Kunugui [voir l'analyse ci-dessus] sur le signe de l'intégrale d'énergie lorsque le noyau satisfait aux hypothèses énumérées. Étude des points-frontière d'un compact qui satisfont à la condition de Poincaré généralisée (moyennant une hypothèse de régularité supplémentaire sur le noyau). *J. Deny.*

Slivnyak, I. M. On the uniqueness theorem in the theory of the potential. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 473–491 (1950). (Russian)

L'auteur donne dans le plan un exemple de distribution de masses (non identiquement nulle) sur un compact sans intérieur et sans points instables (c'est-à-dire sans points d'effilement du complémentaire), telle que le potentiel logarithmique soit constant sur chaque domaine composant δ_i du complémentaire. Il rappelle que cette question est liée aussi à celle de l'approximation d'une fonction continue sur un compact par certaines fonctions harmoniques. La complication de l'exemple, formé avec une infinité de couronnes dont les circonférences portent des masses, n'est pas inattendue lorsque l'on sait avec le rapporteur [Ann. Inst. Fourier Grenoble **1** (1949), 113–120 (1950); ces Rev. **12**, 258] qu'un tel exemple (aussi dans l'espace) serait impossible avec un nombre fini de domaines du complémentaire, chacun n'étant effilé en aucun de ses points-frontières.

M. Brelot (Grenoble).

Smolickiĭ, H. L. Estimates of the derivatives of fundamental functions. Doklady Akad. Nauk SSSR (N.S.) **74**, 205–208 (1950). (Russian)

A surface S belongs to Λ_k if there exist positive constants d, A such that for every point P on S the part of S lying within the sphere of radius d around P admits (in a properly chosen Cartesian coordinate system with P as origin) the representation $\zeta=\omega(\xi,\eta)$, with $\zeta_\xi(0,0)=\zeta_\eta(0,0)=0$, and ω possessing continuous partial derivatives up to the order k, the moduli of which do not exceed A. Let S bound a domain Ω, and let $v_1, v_2, \cdots$; $\lambda_1, \lambda_2, \cdots$ be the eigenfunctions and eigenvalues of the equation $\Delta v+\lambda^2 v=0$ for the boundary condition $v=0$ on S. The author proves that if $S\in\Lambda_{l+3}$, then for $2k+1<l$ the kth order partial derivatives $D^k v_m$ satisfy the inequalities $|D^k v_m|<A_k\lambda_m^{2k+2}$, A_k depending only on Ω. The proof is based on potential-theoretical results refining known relations between Hölder-continuity and differentiability of densities and their potentials. *L. Bers.*

***Sanderson, Judson, Jr. Study of the Generalized Potential Integral.** Abstract of a Thesis, University of Illinois, 1950. i+3 pp.

"The purpose of the paper is the study of integrals of the form $I=\int\int\cdots\int v\,d\mu/R(\alpha,p)$, where V is a bounded n-dimensional Euclidean space, μ is an additive function of Lebesgue measurable sets, and $R(\alpha, p)$ is the distance between α and the variable point of integration, p."

Extract from the paper.

Warschawski, S. E. On the Green function of a star-shaped three dimensional region. Amer. Math. Monthly **57**, 471–473 (1950).

The author gives elegant and brief proofs of two results due to Gergen [Amer. J. Math. **53**, 746–752 (1931)] as follows: Let D be a domain in three space which is star-shaped with respect to the origin. Let g be the Green's function of D with pole at the origin. Then (1) the level curves of g are star-shaped with respect to the origin, and (2) $|\text{grad } g|^2\leqq g^2/r^2$, the equality holding only when $g\equiv 1/r$. The proof is made by applying the principle of the maximum to the function $h(P)=[g(P)-\lambda g(\lambda P)]/(1-\lambda)$, where $0<\lambda<1$. The function $h(P)$ is harmonic in D, has nonpositive boundary limits, and so is nonpositive. This immediately translates into (1), and (2) follows by considering the result of letting $\lambda\to 1$. The result is shown to be valid for the extended Green's function in case the ordinary one does not exist. *J. W. Green* (Los Angeles, Calif.).

Taylor, A. E. A note on the Poisson kernel. Amer. Math. Monthly **57**, 478–479 (1950).

Graffi, Dario. Sull'elettrostatica dei conduttori non omogenei. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **6**, 197–204 (1950).

The author considers a system of n conductors in a dielectric. All of the conductors but one are homogeneous, and that one is composed of two homogeneous parts. The object is to determine the relations between the charges on the various conductors and the potentials thereon. These relations are obtained by applying Gauss's theorem to the various conductor potentials associated with the problem, and involve the capacity of each conductor, the mutual capacities, and n other constants. These other constants are given in terms of surface integrals of that harmonic function which vanishes on the surfaces of the homogeneous conductors and on one of the parts of the nonhomogeneous one, and which is equal to one on the other part. A number of inequalities and estimates relating these constants to the capacities are obtained. *J. W. Green.*

Lenz, Friedrich. Annäherung von rotationssymmetrischen Potentialfeldern mit zylindrischen Äquipotentialflächen durch eine analytische Funktion. Ann. Physik (6) **8**, 124–128 (1950).

Differential Equations

Karapandzitch, Georges. Sur une application des intégrales singulières des équations différentielles ordinaires. Bull. Soc. Math. Phys. Serbie **2**, nos. 1–2, 37–47 (1950). (Serbo-Croatian. French summary)

To obtain the primitive of $y(x)$ write $\int y\,dx=xy-\int x\,dy$, solve $y=y(x)$ for x and find $\int x\,dy$. This device seems to escape the author and he discovers it as a new method by interpreting $\int y\,dx$ as the singular integral of a Clairaut equation. He resorts to the singular integral of a generalized Clairaut equation to obtain $\int y(\psi)\psi'\,dx$. *M. Golomb.*

Šilov, G. E. Integral curves of a homogeneous equation of the first order. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 193–203 (1950). (Russian)

The author gives a simple and elegant discussion of the equation $y'=f(y/x)$ with emphasis on the qualitative nature

of its solutions. The starting point for the exposition is the observation that if k_0 satisfies the equation $k_0=f(k_0)$, then the ray $y=k_0x$ is an integral curve. If no such k_0 exists, then the integral curves are all either spirals or closed curves about the origin; indeed, for a precise determination it is only necessary to follow one curve through one circuit about $(0, 0)$, since the differential equation is unchanged by the transformation $(x, y)\rightarrow(ax, ay)$. If integral rays exist, then it is necessary to investigate the behavior of nearby solutions; the behavior depends on the value of $f'(k_0)$, with three different cases arising according as $f'(k_0)>1$, $1>f'(k_0)>0$, or $0>f'(k_0)$. The method is applied to the classical case $f(y/x)=(ax+by)/(cx+dy)$, and Poincaré's classification of the types of singularities possible at the origin is easily obtained. Some more complicated cases are also discussed, and an error in the work of Forster [Math. Z. 43, 271–320 (1937)] is corrected. *J. G. Wendel* (New Haven, Conn.).

Slivinskiĭ, V. E. On an application of the generalized theory of functions of a complex variable. Doklady Akad. Nauk SSSR (N.S.) 74, 897–899 (1950). (Russian)

N. P. Erugin has discussed [Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 315 (1950); these Rev. 12, 99] the system of ordinary differential equations $dx/dt=u(x, y)$, $dy/dt=v(x, y)$, where $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann partial differential equations. Generalizing the above problem, the author studies the set of equations $dx_j/dt=u_j(x_0, x_1, \cdots, x_{n-1})$, $j=0, 1, \cdots, n-1$, where the u_j satisfy the conditions of monogeneity of N. M. Krylov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 683–684 (1947); these Rev. 9, 233]. The quadrature is given in terms of generalized monogenic functions; the computations for one special case are given in detail. *E. F. Beckenbach* (Los Angeles, Calif.).

Duff, G. F. D. Limit cycles of systems of the second order. Proc. Nat. Acad. Sci. U. S. A. 36, 749–752 (1950).

The limit cycles of the system $\dot{x}=P(x, y)$, $\dot{y}=Q(x, y)$ are studied by the device of imbedding this system in what the author calls a "complete family" of systems $\dot{x}=P(x, y, \alpha)$, $\dot{y}=Q(x, y, \alpha)$. Such a family can essentially be characterised by the properties that $P(x, y, \alpha+\pi)=-P(x, y, \alpha)$, $Q(x, y, \alpha+\pi)=-Q(x, y, \alpha)$, and that the vector with components P, Q turns in the positive sense as α increases. The following theorems concerning complete families are stated. (1) Limit cycles corresponding to different values of α do not intersect. (2) Limit cycles which are completely stable or completely unstable expand or contract monotonically as α varies in a fixed sense, while semistable cycles either split into two diverging cycles or disappear. (3) Each elementary critical point of positive determinant either generates or absorbs exactly one cycle in the range $0\leqq\alpha\leqq\pi$. It is then shown how these theorems and some of their consequences can be used to gain information on the location, or on the nonoccurrence, of limit cycles in certain regions of the (x, y)-plane. The proofs are only partly sketched.

W. Wasow (Cambridge, Mass.).

Erugin, N. P. A qualitative investigation of the integral curves of a system of differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 659–664 (1950). (Russian)

Erugin, N. P. On certain questions of stability of motion and the qualitative theory of differential equations in the large. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 459–512 (1950). (Russian)

The two papers under review deal with the following problem proposed by M. A. Aĭzerman [Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 187–188 (1949); these Rev. 11, 177]: Consider the following two systems of the nth order, the first with constant coefficients,

$$(1)\qquad \frac{dx_1}{dt}=\sum a_{1j}x_j+ax_k;\qquad \frac{dx_i}{dt}=\sum a_{ij}x_j,\qquad i>1,$$

$$(2)\qquad \frac{dx_1}{dt}=\sum a_{1j}x_j+f(x_k);\qquad \frac{dx_i}{dt}=\sum a_{ij}x_j,\qquad i>1.$$

Suppose that the a_{ij} are such that when a varies in a certain interval $\alpha<a<\beta$ the characteristic roots of (1) all have negative real parts, i.e., the origin is stable. To prove or disprove that, whatever the continuous single-valued function $f(x)$ such that $\alpha x^2<xf(x)<\beta x^2$, $x\neq0$, $f(0)=0$, every integral curve of (2) tends to the origin as $t\rightarrow+\infty$. The author discusses very extensively the case $n=2$ (with $k=1$), verifying the conjecture for certain intervals. The scope of his results may be indicated by the following special case: If $a_{22}\neq0$ and $\epsilon=\pm1$ has the sign of a_{22} and if (3) $-\epsilon^3\Delta/a_{22}<\epsilon xf(x)>-(a_{11}+a_{22})x^2$, $x\neq0$, (4) $\Delta=a_{11}a_{22}-a_{12}a_{21}>0$, $a_{11}+a_{22}<0$, then every integral curve of (2) tends to the origin as $t\rightarrow+\infty$. (The inequalities (4) merely assert that for (1) and $a=0$ the origin is stable.) By and large the author's results are obtained by direct examination of the general disposition of the integral curves in the plane. *S. Lefschetz* (Princeton, N. J.).

Saharnikov, N. A. On conditions for the existence of a center and a focus. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 513–526 (1950). (Russian)

The author treats the system of equations,

$$dx/dt=y+f(x, y),\quad dy/dt=-x+g(x, y),$$

where f and g have power series expansions in x and y in the neighborhood of the origin, and considers the problem of determining when the origin is a center or a focus. The results are obtained by considering the related partial differential equation $(y+f)\partial F/\partial x+(g-y)\partial F/\partial y=0$.

R. Bellman (Stanford University, Calif.).

Vasil'eva, A. B. On differentiation of solutions of systems of differential equations containing a small parameter. Doklady Akad. Nauk SSSR (N.S.) 75, 483–486 (1950). (Russian)

Continuing her earlier work [same Doklady (N.S.) 61, 597–599 (1948); these Rev. 10, 298] the author investigates the solutions of the system $\dot{y}=f(t, y, z)$, $\mu\dot{z}=F(t, y, z)$; μ is a small positive parameter, and y, z are vectors, not necessarily of the same dimension. The theorem states that (under rather complicated conditions) the limits of $\partial y/\partial\mu$, $\partial z/\partial\mu$ as $\mu\rightarrow0^+$ exist and satisfy certain natural relations. The hypotheses on the functions f, F, and on the initial conditions are essentially those of Tihonov [Mat. Sbornik N.S. 27(69), 147–156 (1950); these Rev. 12, 181].

J. G. Wendel (New Haven, Conn.).

Levinson, Norman. On stability of non-linear systems of differential equations. Colloquium Math. 2, 40–45 (1949).

The author considers the Poincaré-Liapounoff stability of the nonlinear system of differential equations

$$dz/dt=Az+f(z, dz/dt, t).$$

This problem was discussed by the reviewer [Trans. Amer. Math. Soc. 62, 357–386 (1947); these Rev. 9, 436] under certain restrictions on $\partial f/\partial z$, $\partial f/\partial w$, where $f=f(z, w, t)$. The

requirement that the partial derivatives exist is removed in the present paper, and very simple proofs are given in the two cases where all the real parts of the characteristic roots of A are negative, and where all the solutions of $dy/dt=Ay$ are bounded. *R. Bellman.*

Yakubovič, V. A. On the boundedness of the solutions of the equation $y''+p(t)y=0$, $p(t+\omega)=p(t)$. Doklady Akad. Nauk SSSR (N.S.) **74**, 901–903 (1950). (Russian)

Considering the vector-matrix differential equation $df/dt=A(t)f$, where $A(t)$ is a continuous periodic matrix whose trace is zero, the author deduces a correspondence between various sets in the space of matrices possessing the above properties and certain intervals of the t-axis. The sets of matrices are determined by various properties of the solutions such as periodicity, boundedness, and so on. The intervals are determined from certain oscillatory properties of the solutions. These results are used to deduce criteria for boundedness of the solutions of $u''+p(t)u=0$, where $p(t)$ is periodic and continuous. These criteria are analogous to those given by Borg [Ark. Mat. Astr. Fys. **31A**, no. 1 (1944); these Rev. **8**, 70]. *R. Bellman.*

Friedlander, F. G. On the forced vibrations of quasi-linear systems. Quart. J. Mech. Appl. Math. **3**, 364–376 (1950).

In the differential equation $\ddot{x}+\omega^2x=kf(x, \dot{x}, t)$ let ω be a constant and k a small parameter. The function $f(x, \dot{x}, t)$ has the period 2π in t. The "amplitude" $r(t)$ and the "phase" $\theta(t)$ of a solution $x(t)$ are defined by the relations $x=r\sin\theta$, $\dot{x}=\omega r\cos\theta$. The author studies the asymptotic behavior of r and θ as $t\to\infty$, using a method that is related to a procedure of Kryloff and Bogoliuboff [Introduction to Non-Linear Mechanics, Princeton University Press, 1943; these Rev. **4**, 142]. The case studied in greatest detail is that in which $f(x, \dot{x}, t)$ is a polynomial in x, $\dot{x}$, $\cos t$, $\sin t$, and in which ω does not belong to a certain finite "critical set" of rational numbers determined by $f(x, \dot{x}, t)$. In this case there is shown to exist a finite set of positive numbers, $a_1, \cdots, a_n$, such that $r(t)$ will, for sufficiently large t, remain in an interval of width $O(k)$ about one of the a_i. If $f(x, \dot{x}, t)$ is not of that special form, a less precise form of these results remains true. Analogous investigations of the phase $\theta(t)$ lead to some necessary conditions for the existence of subharmonic solutions. If ω is a function of k whose difference from a number $\omega_0=p/q$ of the critical set is $O(k)$, it is proved that the previous results subsist, provided p and q are large or $k/|\omega-\omega_0|$ is small. *W. Wasow.*

Minorsky, Nicolas. Sur l'excitation paramétrique. C. R. Acad. Sci. Paris **231**, 1417–1419 (1950).

The author studies the asymptotic behavior, for large t, of the solutions of the differential equation

$$\ddot{x}+p\dot{x}+(1+\gamma\cos 2t)x+\epsilon x^3=0,$$

where γ, p, and ϵ are small parameters. If $\epsilon=0$, the result is that the amplitude $r=(x^2+\dot{x}^2)^{\frac{1}{2}}$ tends to infinity, while the phase $\phi=\arctan(\dot{x}/x)$ approaches the limit $\frac{1}{4}\pi$. For $\epsilon\neq 0$, both r and ϕ approach finite limits, as $t\to\infty$, provided γ and p are of the order of magnitude of ϵ, and $\gamma<2p$. No complete proofs are given. *W. Wasow.*

Meerov, M. V. On systems of autoregulation stabilized for an arbitrarily large coefficient of amplification. Avtomatika i Telemehanika **8**, 225–242 (1947). (Russian)

In this paper a general method is considered for the synthesis of linear control mechanisms which are to be stabilized for arbitrarily large amplification coefficients by the use of a prescribed type of stabilizing loops. Obtained are general rules for connecting a stabilizing loop of the prescribed type, the rules depending on the number of elementary links in the system. They are derived from conditions under which the polynomial $P_1(p)+mP_2(p)$ is stable (that is, all its zeros have negative real part) for all sufficiently small m, when it is known that $P_1(P)$ is a stable polynomial. *M. Golomb* (Lafeyette, Ind.).

Gol'dfarb, L. S. On some nonlinearities in systems of regulation. Avtomatika i Telemehanika **8**, 349–383 (1947). (Russian)

Self-oscillations in automatic control systems containing various nonlinear elements, their frequencies, amplitudes, and stability are determined by the method of approximative "harmonic balance." The approximation is of the first order, being essentially the well-known method of "linearization" [see, e.g., Kryloff and Bogoliuboff, Introduction to Non-Linear Mechanics, Princeton University Press, 1943; these Rev. **4**, 142]. Numerous concrete regulating schemes are treated. *M. Golomb* (Lafayette, Ind.).

Andronov, A., and Maier, A. Vyšnegradskiĭ's problem in the theory of direct regulation. I. The theory of the regulator of direct action in the presence of Coulomb and viscous friction. Avtomatika i Telemehanika **8**, 314–334 (1947). (Russian)

The dynamics of Watt's governor is one of the oldest and most extensively studied problems in the theory of automatic control mechanisms. In this paper the authors present the first part of a careful analytic treatment of the direct centrifugal regulator (driven without servomotor) under the usual simplifying assumptions in the presence of Coulomb and viscous friction. The equations, going back to Vyšnegradskiĭ [C. R. Acad. Sci. Paris **83**, 318–321 (1876)], are $d^2x/dt^2+Bdx/dt+Ax=y\mp\frac{1}{2}$, $dy/dt=-x$. The portrait of the corresponding phase space for various values of the parameters is analyzed by the classical methods of Liapounoff and Poincaré. *M. Golomb* (Lafayette, Ind.).

Kazakevič, V. V. Multiply valued systems and the simplest dynamical models of clocks. Doklady Akad. Nauk SSSR. (N.S.) **74**, 665–668 (1950). (Russian)

Multiply valued systems arise in connection with certain types of clocks and various regulators with elements which have characteristics à la hysteresis. The general type of differential equations arising with a single degree of freedom is $\ddot{x}=F(x, \dot{x})$, where: (a) There is an attached finite open covering of the plane by regions G_i; (b) in each G_i the function F is equal to a certain $f_i(x, \dot{x})$ which satisfies a Lipschitz condition in the region. Related multiply sheeted surfaces are discussed. An application is made to a clock with escapement and friction, the surface being then two sheeted. *S. Lefschetz* (Princeton, N. J.).

Bulgakov, B. V. Regulating circuits with links having up to several degrees of freedom. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 619–634 (1950). (Russian)

The regulating circuits discussed are constructed from units connected together in a circle. The units are governed by linear differential equations with constant coefficients. Applications are made to problems involving a ship steering mechanism and a gyroscopic stabilizer. *E. N. Gilbert.*

Šul'gin, M. F. Reduction of systems of differential equations to the form of Lagrange. Doklady Akad. Nauk SSSR (N.S.) **75**, 349–351 (1950). (Russian)

If $\bar{q}_k = f_k(t; q_i; \dot{q}_i)$, $i, k = 1, \cdots, n$, and

$$L = \sum_i \dot{q}_i \dot{q}_{n+i} + \sum_i f_i \dot{q}_{n+i},$$

the last n Lagrange equations derived from L coincide with the original system, while the first n define the q_{n+i}. In the special case when f_k does not contain any q_i, all the q_i are cyclic variables in L and Routh's reduction results in a Lagrangian system [derived from $R = L - \sum \dot{q}_i(\partial L/\partial \dot{q}_i)$] in q_{n+i} alone. A particular solution of the latter yields the general solution of the original system in the form $q_i = -\int (\partial R/\partial c_i)\, dt$, $c_i = \delta_i$. *A. W. Wundheiler.*

Breves Filho, J. A. On the algebraic integrals of a system of differential equations of mechanics. Proc. Amer. Math. Soc. **1**, 498–505 (1950).

È noto che le equazioni differenziali del moto di un corpo rigido pesante fissato in un punto O nel caso in cui l'ellissoide di inerzia relativo ad O è rotondo e il centro di massa appartiene al piano equatoriale dello stesso ellissoide non ammette integrali algebrici oltre quelli classici. La dimostrazione di questa proprietà è stata iniziata da R. Liouville [Acta Math. **20**, 239–284 (1896), pp. 244–247] e continuata da E. Husson [ibid. **31**, 71–88 (1907)] e P. Burgatti [Rend. Circ. Mat. Palermo **29**, 369–377 (1910)]. L'autore osserva che tale dimostrazione è incompleta e ne espone una nuova partendo dal sistema differenziale $(S; \lambda, \rho)$ nella forma seguente:

$$dx_1/dt = -\tfrac{1}{2}\lambda x_1 x_3 + y_3, \quad dx_2/dt = \tfrac{1}{2}\lambda x_2 x_3 - \rho y_3,$$
$$dx_3/dt = (1/(2-\lambda))(\rho y_1 - y_2);$$
$$dy_1/dt = \rho x_1 y_3 - x_1 y_1, \quad dy_2/dt = x_1 y_2 - \rho x_1 y_3,$$
$$dy_3/dt = \tfrac{1}{2}(x_2 y_1 - x_1 y_2),$$

e considerando in particolare il caso $\rho = 1$. Le equazioni restano inalterate quando si moltiplicano rispettivamente le x, le y, e la t per le potenze k, k^2, k^{-1} di una costante k. Ogni funzione $f(x, y)$ delle variabli $x_1, x_2, x_3, y_1, y_2, y_3$ è detta omogenea di grado m se vale l'identità $f(kx, k^2 y) = k^m f(x, y)$. L'autore dimostra i due teoremi seguenti: (1) Ogni integrale omogeneo e razionale del sistema differenziale $(S; \lambda, 1)$ è il quoziente di due integrali interi ed omogenei. (2) Il sistema differenziale $(S; 1, 1)$ non ammette integrali algebrici oltre i seguenti:

$$x_1 x_2 + \tfrac{1}{2}x_3^2 + y_1 + y_2 = h_1, \quad x_1 y_2 + x_2 y_1 + x_3 y_3 = h_2,$$
$$y_1 y_2 + y_3^2 = h_3, \quad (x_1^2 - 2y_1)(x_2^2 - 2y_2) = h_4$$

che si riferiscono al caso della Kowalevsky [Acta Math. **12**, 177–232 (1889)].

G. Lampariello (Messina).

Popov, B. S. Sur une condition d'intégrabilité de d'Alembert relative à l'équation différentielle de la balistique. Bull. Soc. Math. Phys. Macédoine **1**, 29–39 (1950). (Macedonian. French summary)

The fundamental equation $(y+\rho)y' + y^2 - 1 = 0$ may be transformed in the particular case where $\rho = Ax^2 + Bx + C$ (A, B, C constants) into a differential equation of hypergeometric functions. Four known cases of integrability may be reduced to a single condition $B^2 = A^2/k^2 + 4k^2 + 4AC$ (k an integer), which represents a generalization of that of d'Alembert.

W. S. Jardetzky (New York, N. Y.).

Zátopek, A. Dynamical magnification of a seismograph excited by a shock of the form $\lambda^n e^{-\lambda\tau}\tau^n$. Časopis Pěst. Mat. Fys. **75**, 103–111 (1950). (English. Czech summary)

In a previous paper [Publications du Bureau Central Seismologique International, Série A. Travaux Scientifiques, fascicule **17**, 81–88 (1950)] the author gave a general solution of the damped pendulum equation involving an acceleration of a special type:

$$\frac{d^2y}{d\tau^2} + 2\alpha\frac{dy}{d\tau} + y = -\frac{d^2}{d\tau^2}(\lambda^n e^{-\lambda\tau}\tau^n),$$

under the conditions $0 \leqq \alpha \leqq 1$; and $y = 0$ and $dY/d\tau = 0$ when $\tau = 0$, where y = displacement of the pendulum indicator free from solid friction, Y = ground displacement × static magnification, $Y(\tau) = A\tau^n e^{-\lambda\tau}$, τ = dimensionless time variable $= 2\pi t/T_0$, T_0 = free period of the undamped instrument, A, λ are amplitude constants. The resulting form of forced vibration was discussed for $n = 3$ and various values of the parameters α and λ, respectively. Several sets of curves were plotted. However, the published curves could not be used immediately in the evaluation of seismograms because the parameters were not directly determinable from them. In the present paper the author transforms the curves of dynamic magnification and those of phase lag to a form directly utilizable in the analysis of seismographic records. On p. 104 negative exponential k is written instead of τ, the succeeding parenthesis is not closed, and a superfluous exponent n is applied to (-1) in the equation (4).

J. B. Macelwane (St. Louis, Mo.).

Reiz, Anders. The structure of degenerate stellar configurations near the limiting mass. Astrophys. J. **109**, 303–307 (1949).

The structure of completely degenerate gas spheres in equilibrium under their own gravitation is governed by the equation

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) + \left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2} = 0,$$

containing the single parameter y_0. In this paper the equation is solved by the perturbation method for $1/y_0^2 \leqq 0.01$ and the perturbation function has been calculated and tabulated numerically. From these results are obtained constants and physical characteristics pertaining to the original astronomical problem. *W. E. Milne* (Los Angeles, Calif.).

Viguier, Gabriel. Notions métriques liées à une vibration moléculaire quatrième puissance. Ann. Fac. Sci. Univ. Toulouse (4) **11** (1947), 93–100 (1949).

The wave equation for an anharmonic oscillator with potential function varying as the fourth power, instead of the second power, of the displacement, becomes

$$Y'' + (\lambda_n - x^4)Y = 0, \tag{1}$$

after a change of variables. The approximate formula $\lambda_n = .867(2n+1)^{4/3}$ is found for the eigenvalues. The transformation $y = Y'/Y$ sends (1) into the Riccati equation (2) $y' + y^2 + \lambda_n = x^4$. The author then uses a geometric method to find approximate solutions of the Riccati equation (2).

O. Frink (State College, Pa.).

Hartman, Philip, and Putnam, Calvin R. The gaps in the essential spectra of wave equations. Amer. J. Math. **72**, 849–862 (1950).

This paper contains asymptotic results concerning the eigenvalues and eigenfunctions of the self-adjoint boundary

value problem determined by the equation

(*) $$x''+(\lambda+q)x=0$$

(which is supposed to be of the Grenzpunkt type) and the condition $x(0)\cos\alpha+x'(0)\sin\alpha=0$. The function q is supposed to be continuous and real-valued for $0\leqq t<\infty$. The set S' of cluster points of the spectrum $S(\alpha)$ of this problem is independent of α and is called the essential spectrum of (*). If q is bounded, there exists a positive number $k=k(\lambda)$ such that (**) $x^2+x'^2=O(e^{kt})$ as $t\to\infty$ holds for any solution x of (*). Define $K=K(\lambda)$ as the greatest lower bound of all the numbers k for which (**) holds for every solution of (*). The two main theorems are: (I) For $\delta>0$, let $\omega(t,\delta)=\text{fin sup}\,|q(s)-q(t)|$ for $|s-t|\leqq\delta$, and $\omega(\delta)=\limsup T^{-1}\int_0^T\omega(t,\delta)\,dt$ as $T\to\infty$. There exists a constant C (independent of λ_0) such that, for every sufficiently large λ_0, the interval $\lambda_0\leqq\lambda\leqq\lambda_0+C[\omega(\pi/\lambda_0^{\frac{1}{2}})+\lambda_0^{-1}]$ contains at least one point of the essential spectrum S' of (*); (II) $K(\lambda)=O(\lambda^{-\frac{1}{2}}[\omega(\pi/\lambda^{\frac{1}{2}})+\lambda^{-1}]$ as $\lambda\to\infty$. Somewhat sharper estimates are obtained under the imposition of additional local smoothness conditions on q.

M. J. Gottlieb (Chicago, Ill.).

Kornhauser, Edward, et Stakgold, Ivar. Application du calcul des variations au problème $\Delta u+\lambda u=0$. C. R. Acad. Sci. Paris **232**, 390–391 (1951).

Let any region R in the (x, y)-plane be bounded by a closed curve B and consider the two problems:

(I) $$\Delta u+\lambda u=0, \qquad (u)_B=0;$$
(II) $$\Delta v+\mu v=0, \qquad (\partial v/\partial n)_B=0;$$

in the region R. It is shown that if λ_n $(n=0, 1, 2, \cdots)$ are the proper values of (I) and μ_n $(n=1, 2, 3, \cdots)$ are those of (II), then $\mu_1<\lambda_0$. [That is, in wave guides of constant cross section, the fundamental mode of propagation is that associated with problem (II).] In the course of the proof, the following theorem is also established: of all regions of the same area, the circle yields the largest value of μ_1.

C. G. Maple (Washington, D. C.).

Olevskiĭ, M. N. Triorthogonal systems in spaces of constant curvature in which the equation $\Delta_2 u+\lambda u=0$ allows a complete separation of variables. Mat. Sbornik N.S. **27(69)**, 379–426 (1950). (Russian)

In a three-dimensional Riemann space whose line element has the orthogonal form $ds^2=\sum_{i=1}^{3}H_i(x)\,dx_i^2$, the equation $\Delta_2 u+\lambda u=0$ (Δ_2 is the second differential parameter of Beltrami) is said to permit complete separation of variables if it has solutions of the form $u=u_1(x_1)\cdot u_2(x_2)\cdot u_3(x_3)$, where u_i'/u_i contains at least two independent parameters. In the Euclidean case all triply orthogonal systems which can serve as $x_i=c$, c a constant, are known. They consist of confocal systems of second order surfaces and their degenerate cases (where one or more systems consist of planes). The present paper accomplishes the same for the spaces of constant positive or negative curvature. All triply orthogonal systems with their corresponding line elements and solutions of $\Delta_2 u+\lambda u=0$ are enumerated and given explicitly. Again only systems of confocal second degree surfaces and their degenerations solve the problem, one in the spherical case and seven in the hyperbolic, corresponding to the smaller and greater variety of quadrics, respectively.

H. Busemann (Los Angeles, Calif.).

Gillis, Paul P. Équations de Monge-Ampère à quatre variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 474–484 (1950).

The partial differential equation

(*) $$(z_{11}+z_{22})(z_{33}+z_{44})-(z_{13}+z_{24})^2-(z_{14}-z_{23})^2 = f(x_1, x_2, x_3, x_4), \quad f>0, \quad z_{ij}=\partial^2 z/\partial x_i\partial x_j$$

is said to be of elliptic type with reference to a solution z if $(z_{11}+z_{22})(z_{33}+z_{44})>0$. The author shows that the Dirichlet problem associated with equation (*) has at most two solutions in the class of solutions for which it is of elliptic type.

F. G. Dressel (Durham, N. C.).

Luchak, G., and Langstroth, G. O. Applications of diffusion theory to evaporation from droplets and flat surfaces. Canadian J. Research. Sect. A. **28**, 574–579 (1950).

The boundary value problem $D\partial^2 c/\partial x^2=\partial c/\partial t$, $c(x, 0)=0$, $c[a(t), t]=c_0$ describes the concentration of vapor $c(x, t)$ in the semi-infinite air mass lying above a retreating flat surface $x=a(t)$ at which there is a concentration c_0 (saturation). In contrast to other treatments of the problem, the air is assumed free of vapor initially. Solutions exhibited for this problem, as well as for one requiring zero concentration at an upper absorbing wall, $c(b, t)=0$, call for small values of c_0/ρ, the ratio of densities of vapor and liquid (1.7×10^{-5} for water). When the corresponding problem for spherical droplets is attacked, the authors find it necessary to assume a fixed radius, and obtain an approximate solution $c_1(r, t)$. A second approximation is found from $D\nabla^2 c_2=\partial c_1/\partial t$, with correct boundary conditions reimposed. Further approximations follow in like manner. *R. E. Gaskell.*

Beckert, Herbert. Über quasilineare hyperbolische Systeme partieller Differentialgleichungen erster Ordnung mit zwei unabhängigen Variablen. Das Anfangswertproblem, die gemischte Anfangs-Randwertaufgabe, das charakteristische Problem. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **97**, no. 5, 68 pp. (1950).

The author considers a quasi-linear system of equations of the form

(1) $$\sum_{k=1}^{n}\left(a_{ik}\frac{\partial u_k}{\partial x}+b_{ik}\frac{\partial u_k}{\partial y}\right)=f_i, \qquad (i=1, \cdots, n)$$

where the a_{ik}, b_{ik}, f_i are given functions of x, y, and the u_r. The characteristic matrix is assumed to have real roots corresponding to simple elementary divisors. The first part deals with the Cauchy problem, in which the u_i are prescribed along an arc of a noncharacteristic curve. With the help of the characteristic roots the problem can be reduced (in the small) to the analogous problem for a system of equations of the form

$$\sum_k p_{ik}\left(\alpha_i\frac{\partial u_k}{\partial x}+\frac{\partial u_k}{\partial y}\right)=T_i, \qquad (\alpha_i>0)$$

with initial data prescribed on $x+y=0$. The solution of the latter problem is established by the method of finite differences. (The same problem has been solved by R. Courant and P. Lax using iteration instead of difference methods [Comm. Pure Appl. Math. **2**, 255–273 (1949); these Rev. **11**, 441].) A mixed boundary-initial value problem for a system (1) can be reduced to the solution of Cauchy problems and of problems of a different type, which are taken up in part II of this paper. Here the u_i have to be determined in an angular region between a characteristic curve C and a noncharacteristic curve R. On C the u_i are given, on R a relation of the form $\sum_{i=1}^{m}m_i(x, y, u_k)du_i/ds=m_{n+1}(x, y, u_k)$

is prescribed. This problem is solved again by means of finite differences. [For the solution of a similar problem see M. Cinquini Cibrario, Ann. Scuola Norm. Super. Pisa (3) 3 (1949), 161–197 (1950); these Rev. 12, 337.] *F. John.*

Hodge, P. G., Jr. On the method of characteristics. Amer. Math. Monthly 57, 621–623 (1950).

Sbrana, Francesco. Su un problema di integrazione delle equazioni differenziali lineari alle derivate parziali a coefficienti costanti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 457–462 (1950).

This paper is concerned with the solution of the equation (1) $u_{xx}+au_{yy}+bu_y+cu=0$ $(0\leqq x\leqq l,\ -\infty<y<+\infty)$, subject to the conditions (2) $u(0, y)=\phi(y)$, $u(l, y)=\psi(y)$, $\lim_{y\to\pm\infty} u(x, y)=0$. Here a, b, c are assumed to be real constants and the functions ϕ, ψ shall be representable by their Fourier integrals. The unique solution of the Dirichlet problem is constructed except for the case when there exist bounded solutions of (1) which vanish for $x=0, l$ [as, e.g., for $a=1$, $b=0$, $c>\pi^2/l^2$]. The path of integration in the integral representing the solution depends on the "type" of equation (1) [elliptic, parabolic, or hyperbolic]. If conditions (2) are only imposed for $y>0$, a solution can be constructed assuming prescribed values of $u(x, 0)$ if $a\geqq 0$ and of $u(x, 0)$ and $u_y(x, 0)$ if $a<0$. *F. John.*

Sbrana, Francesco. Su un particolare problema al contorno per le equazioni differenziali lineari alle derivate parziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 45–47 (1950).

In a previous note [see the preceding review] the author gave an expression for the solution $u(x, y)$ of a second order partial differential equation with constant coefficients in the strip $0<x<l$, $-\infty<y<+\infty$, with $u(0, t)$, $u(l, t)$ prescribed. The present note contains the verification of that solution for the case of the equation $u_y=u_{xx}$. *F. John.*

Germay, R. H. J. Sur certains systèmes linéaires d'équations aux dérivées partielles du troisième ordre. II. Ann. Soc. Sci. Bruxelles. Sér. I. 64, 26–30 (1950).

Conclusion of proof of results stated in the first part of this paper [same Ann. Sér. I. 63, 148–154 (1949); these Rev. 12, 30]. *F. John* (Los Angeles, Calif.).

Difference Equations, Special Functional Equations

Milkman, Joseph. Note on the functional equations $f(xy)=f(x)+f(y)$, $f(x^n)=nf(x)$. Proc. Amer. Math. Soc. 1, 505–508 (1950).

The theorems proved are. (1) If $f(m+1)>f(m)$ and $f(mn)=f(m)+f(n)$ hold for all positive integers m and n, then $f(m)=c\ln m$. (2) If $f(x)$ is a monotonically increasing function defined for all positive real numbers x such that $f(x^n)=nf(x)$ for all positive integers n, then $f(x)=k\ln x$. (3) If $f(r)$ is a monotonically increasing function on a set S which is everywhere dense among the positive real numbers and contains all positive integral powers of all members of the set, and $f(r^p)=pf(r)$ for any prime p, then $f(r)=k\ln r$. The proof of (1) is done by reduction to Cauchy's theorem [the only continuous functions satisfying $f(xy)=f(x)+f(y)$ are $f(x)=c\ln x$]. The proofs of (2) and (3) are obvious. *J. Aczél* (Miskolc).

Milkman, Joseph. The logarithmic function is unique. Math. Mag. 24, 11–14 (1950).

The author reduces the solution of (1) $f(x)+g(y)=h(xy)$ to that of (2) $F(x)+F(y)=F(xy)$ and so shows that the only solutions of (1) which are continuous at some point are $k\log ax$, $k\log by$, $k\log abxy$. Generalizations are given. The author was evidently unaware of the extensive literature on (2). *R. P. Boas, Jr.* (Evanston, Ill.).

Hadwiger, H. Studie zu einer Funktionalgleichung. Elemente der Math. 5, 86–88 (1950).

In this paper the author determines the general solution of the functional equation (1) $\Phi(\alpha x)-\alpha\Phi(x)=F(x)$, $x>0$, $0<\alpha<1$. It is required that $F(x)$ have a right-handed derivative $F'(0)$ at $x=0$ and that $\tau>1$ exist such that for $x>0$, $F(x)=F(0)+F'(0)x+O(x^\tau)$. The general solution of (1) is then represented as the sum of a particular solution of (1),

$$\Phi_0(x)=\frac{F(0)}{1-\alpha}+\frac{F'(0)}{\alpha\log\alpha}x\log x-\frac{x}{\alpha}\sum_{\nu=0}^{\infty}\left\{\frac{F(\alpha^\nu x)-F(0)}{\alpha^\nu x}-F'(0)\right\}$$

and the general solution $\varphi(x)$ of (2) $\varphi(\alpha x)-\alpha\varphi(x)=0$, which latter solution, by methods familiar to students of q-difference theory, is represented in the form $\varphi(x)=\alpha^y\omega(x\alpha^{-y})$, $y=[\log x/\log\alpha]$, where $\omega(\vartheta)$ is an arbitrary function defined on $\alpha<\vartheta\leqq 1$. *P. E. Guenther* (Cleveland, Ohio).

Fernandez Biarge, Julio. Recurrence equations. Gaceta Mat. (1) 2, 157–165 (1950). (Spanish)

The solutions of linear recurrence equations (i.e., linear difference equations with constant coefficients and variable an integer) by means of the operator Ω, where $\Omega f(x)=f(x+1)$. The results are all well known. *L. M. Milne-Thomson.*

Truesdell, C. On the addition and multiplication theorems for special functions. Proc. Nat. Acad. Sci. U. S. A. 36, 752–755 (1950).

For the solutions of the functional equation

$$\partial F(z, \alpha)/\partial z=F(z, \alpha+1)$$

one has the multiplication theorem

$$(1)\qquad F(kz, \alpha)=\sum_{n=0}^{\infty}\frac{1}{n!}(k-1)^n z^n F(z, \alpha+n).$$

The author points out that most of the known multiplication theorems of special functions are particular instances of (1). *A. Erdélyi* (Pasadena, Calif.).

Robinson, Lewis Bayard. Introduction to a study of a type of functional differential and functional integral equations. Math. Mag. 23, 183–188 (1950).

The functional equation $u'(x)=\lambda u(x^2)/(1-x^2)$, also treated elsewhere by the author [e.g., Revista Ci., Lima 50, 141–150 (1948); these Rev. 10, 541], is shown to have a solution that is entire in λ for all x not a zero of any equation $x^{2^r}=1$, $r=1, 2, \cdots$. This solution is discussed. *I. M. Sheffer.*

Myškis, A. D. Supplementary bibliographical material to the paper "General theory of differential equations with retarded argument." Uspehi Matem. Nauk (N.S.) 5, no. 2(36), 148–154 (1950). (Russian)

The paper referred to in the title appeared in the same journal (N.S.) 4, no. 5(33), 99–141 (1949); these Rev. 11, 365. *W. J. Trjitzinsky* (Urbana, Ill.).

Boas, Ralph P., Jr. Sur une équation fonctionnelle. Elemente der Math. 5, 85–86 (1950).

The author considers the functional equation

$$\int_{-\alpha}^{\alpha} f(x+t)\sin t\,dt=0, \tag{1}$$

α constant, and demonstrates for every $\alpha>0$ the existence of an infinite number of linearly independent solutions. Sets of solutions of the form $e^{\lambda t}$, λ complex and not zero, are first found for each $\alpha>0$ for the equation

$$\int_{-\alpha}^{\alpha} e^{\lambda t}\sin t\,dt=0, \tag{2}$$

where λ is any one of the infinite set of values satisfying $\varphi(\lambda)=\lambda\sin\alpha\cosh\lambda\alpha-\cos\alpha\sinh\lambda\alpha=0$. Every such solution of (2) with $\lambda=\mu+i\nu$ furnishes two real, linearly independent solutions of (1), $e^{\mu t}\cos\nu t$, $e^{\mu t}\sin\nu t$. Further, if $\{\lambda_n\}$ is the sequence of characteristic values for a given α and $\{a_n\}$ a sequence of constants such that $g(t)=\sum_{n=1}^{\infty}a_n e^{\lambda_n t}$ converges uniformly in every bounded domain of the complex plane which contains a real interval, the real and imaginary parts of $g(t)$ furnish new solutions of (1). It is also remarked that, if $f(t)$ is an analytic function, (1) is equivalent to the system of equations $\int_{-\alpha}^{\alpha}f^{(k)}(t)\sin t\,dt=0$, $k=1, 2, \cdots$. Reference is made to earlier consideration of the problem by E. Trost [same journal 2, 76–80 (1947); Intermédiaire Recherches Math. 4, 99 (1948); these Rev. 9, 54]. *P. E. Guenther* (Cleveland, Ohio).

Functional Analysis, Ergodic Theory

Köthe, Gottfried. Über zwei Sätze von Banach. Math. Z. 53, 203–209 (1950).

Banach has shown that a linear continuous map of an (F)-space (linear, metric, complete space) onto another (F)-space takes open sets into open sets. Using essentially the same method of proof, the author obtains this result for spaces of the following type: E is the linear extension of locally convex (F)-spaces $E_1\subset E_2\subset\cdots$; the topology T_{n+1} of E_{n+1} is, relative to E_n, not finer than T_n; a generic neighborhood of the origin in E is the absolute-convex hull of the U_n ($n=1, 2, \cdots$), where U_n is a neighborhood of the origin in E_n. The second part of the paper is concerned with generalizations of another theorem due to Banach to the effect that a linear subspace H of the conjugate space E' is weakly closed if and only if the intersection of H with bounded closed sets is compact. For Stufenräume, the author gives a new proof that a subset is T_k-closed if and only if it is weakly sequentially closed. For the notation as well as the original proof, see a previous article by the author [Math. Z. 51, 317–345 (1948); these Rev. 10, 255]. *R. S. Phillips* (Princeton, N. J.).

Köthe, Gottfried. Über die Vollständigkeit einer Klasse lokalkonvexer Räume. Math. Z. 52, 627–630 (1950).

Let $E_1\subset E_2\subset\cdots$ be a strictly increasing sequence of locally convex, complex linear topological spaces, such that the topology T_{n+1} of E_{n+1} induces in E_n a topology not finer than T_n. If E is the union of all E_n, the finest locally convex topology T of E which induces in each E_n a topology not finer than T_n has for a neighborhood basis of 0 the family of sets $\Gamma(W_1, W_2, \cdots)$, where each W_n is a T_n-neighborhood of 0 in E_n and Γ means the absolute-convex hull. Theorem. If a separated locally convex E is constructed as above from a sequence $\{E_n\}$ of separated locally convex spaces, then E is complete (in the sense that every Cauchy filter has a limit) if and only if for each n every Cauchy filter with a basis consisting of sets of E_n has a limit in some E_{n+k}. This theorem implies a result of Dieudonné and Schwartz [see the paper reviewed below, corollary of theorem 6] that every space of type ($\mathfrak{LF}$) is complete; a space of type ($\mathfrak{LF}$) is constructed as above with each E_n a complete, locally convex metrizable linear space such that T_{n+1} imposes on E_n exactly the topology T_n. *M. M. Day.*

Dieudonné, Jean, et Schwartz, Laurent. La dualité dans les espaces ($\mathfrak{F}$) et ($\mathfrak{LF}$). Ann. Inst. Fourier Grenoble 1 (1949), 61–101 (1950).

The authors generalize to some locally convex topological vector spaces known properties of the relationship between strong and weak topologies and of the duality theory for Banach spaces which turn out to be of importance for the newborn theory of distributions [L. Schwartz, Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 57–74 (1946); 23, 7–24 (1948); Théorie des distributions, v. 1, Actualités Sci. Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9, Hermann, Paris, 1950; these Rev. 8, 264; 10, 36; 12, 31]. The main novelty of this article is the systematic consideration of ($\mathfrak{LF}$)-spaces which are built from the ($\mathfrak{F}$)-spaces by a process of inductive sequential limit. An ($\mathfrak{F}$)-space is defined to be a locally convex and complete Hausdorff topological vector space with a countable base of neighborhoods at the origin [notice the slight difference from the terminology of Banach due to the requirement of local convexity]. An ($\mathfrak{LF}$) is defined as follows. Consider a real or complex vector space E and a strictly increasing sequence of vector subspaces E_n, $n=1, 2, \cdots$, whose union is equal to E. Each E_n is assumed to be an ($\mathfrak{F}$)-space. Let this sequence be compatible in the sense that every E_n is a topological subspace of E_{n+1}. Then there exists a well defined topology on E having the largest possible collection of open sets among those topologies which enjoy the two following properties: (1) E is a locally convex topological vector space; and (2) each E_n is a topological subspace of E. Endowed with this topology E is said to be an ($\mathfrak{LF}$)-space. Such spaces occur in several places in functional analysis and allow one to make use of the general methods of the theory of topological vector spaces in situations where only the old-fashioned sequential methods seemed to be available. After studying bounded sets in ($\mathfrak{F}$) and ($\mathfrak{LF}$) spaces, the authors extend to these spaces the most important features of the classic duality theory for normed vector spaces. More specifically, the open mapping theorem, the polarity between bounded sets and neighborhoods at the origin in the space and its dual, the condition for reflexiveness and the duality for subspaces and quotient spaces are here discussed among other topics. An ($\mathfrak{LF}$) is shown to be always complete in Weil's sense [cf. the preceding review]. The compactness theorems of Šmulian [Rec. Math. [Mat. Sbornik] N.S. 7(49), 425–448 (1940); these Rev. 2, 102] and of Eberlein [Proc. Nat. Acad. Sci. U. S. A. 33, 51–53 (1947); these Rev. 9, 42] are also generalized to ($\mathfrak{F}$) and ($\mathfrak{LF}$) spaces. The paper ends with a discussion of the continuity of bilinear functions and a list of fourteen open questions [see the following review]. *L. Nachbin* (Rio de Janeiro).

Grothendieck, Alexandre. Quelques résultats relatifs à la dualité dans les espaces ($\mathfrak{F}$). C. R. Acad. Sci. Paris 230, 1561–1563 (1950).

Consider an ($\mathfrak{F}$) space E. Let E' be its dual endowed with the strong topology and E'' the dual of E' also endowed

with its strong topology [the terminology being that of the paper reviewed above]. The results of the author contain a partial answer to some questions left open by Dieudonné and Schwartz [see the preceding review]. As the author states, every countable bounded set of E'' is contained in the closure (for the weak topology defined by E' on E'') of some bounded set of E. This implies that E'' is strongly complete. Every linear functional on E' which is bounded on bounded sets belongs to the closure (for the weak topology defined by E' on the algebraic dual of E') of some bounded set of E''. If E' is strongly separable, then every bounded subset of E'' is contained in the closure (for the weak topology defined on E'' by E') of some bounded set of E. These and other results are given without proof.

L. Nachbin (Rio de Janeiro).

Alexiewicz, A. On sequences of operations. I. Studia Math. 11, 1–30 (1949).

The author considers continuous operations from a complete metric space X into a metric space Y. He proves that if a sequence of such operations is convergent in a residual set (i.e., the complement of a set of the first category), then it is equicontinuous in a residual set. He also shews that the same result holds with "set of the second category" in place of "residual set" in both hypothesis and conclusion. He then considers classes of operations such that every sequence of operations of the class possesses one or more of the following properties, namely, (r_1) equicontinuity in a residual set implies equicontinuity everywhere, (r_2) equicontinuity in a set of the second category implies equicontinuity everywhere, (r_3) equicontinuity in a set of the second category implies uniform equicontinuity, and (c) convergence in a sphere implies convergence everywhere. He proves a number of results, of which an example is the following: If Y is complete, and if a sequence of operations belonging to a (cr_3)-class converges in a set of the second category, then it converges everywhere and its limit is uniformly continuous. He then applies his results to various special cases, including inter alia (i) Y a Banach space; (ii) Y a space of type (F) [Banach, Théorie des opérations linéaires, Warsaw, 1932, p. 35]; (iii) X a pseudogroup of Saks, Y a space of type (F).

The author wishes me to draw attention to three misprints on p. 4, where "convergence" should read "equicontinuity" in each of (r_1), (r_2), and (r_3). These errors will be listed in the Errata to vol. 11 of Studia Mathematica. [There is also a suffix n omitted in the first line of (1.1) (also on p. 4). Z (on p. 5) should (in the reviewer's opinion) be the space of all sequences of elements of the space Y, else $W(x)$ is not defined everywhere in X. In reference 2 the volume should be 3 (1922) and the final page number 181. In the same reference "leur application" should be singular.]

A. F. Ruston (London).

Orlicz, W. Linear operations in Saks spaces. I. Studia Math. 11, 237–272 (1950).

Let X be a normed linear space with norm $\|\cdots\|$ (not necessarily complete), R the unit sphere of X, and $\|\cdots\|^*$ a possibly different norm in X. In R define a metric $d(x, y)$ by $\|x-y\|^*$. If in this metric, R is complete, we call it a Saks space X_s. The general plan behind Saks spaces is this. The norm $\|\cdots\|^*$ has greater smoothness than the given norm $\|\cdots\|$. For example, if X is the space M of essentially bounded functions on $[0, 1]$, then $\|x(t)\| = \text{ess sup}_{[0,1]} |x(t)|$, and $\|x(t)\|^* = \int_0^1 |x(t)|\, dt$, etc. The following additional hypotheses concerning X_s are often satisfied and prove useful: (Σ_1): Let $K(x, \rho)$ be the open sphere, center x, radius ρ in X_s. Then for each $x \in X_s$, $\rho > 0$, there is a $\delta > 0$ such that $K(0, \delta) \subset K(x, \rho) - K(x, \rho)$. (Σ_2): If $x_n \to 0$ in X_s, if $\epsilon_n > 0$, $\epsilon_n \to 0$, then there is a sequence $k_n < k_{n+1}$ and a sequence $\hat{x}_{k_n}$ such that (i) $d(\hat{x}_{k_n}, x_{k_n}) < \epsilon_{k_n}$, (ii) $\sum_{n=1}^{m} \lambda_n \hat{x}_{k_n} \in X_s$ if $\lambda_n = 0$ or 1, (iii) if $\lambda = \{\lambda_i\}$, $\lambda_i = 0$ or 1, there is an element $\hat{x}_\lambda$ in X_s such that $\sum_{n=1}^{m} \lambda_n \hat{x}_{k_n} \to \hat{x}_\lambda$. (Σ_2'): Let $\epsilon(x) \geqq 0$, $\epsilon(x) \to 0$ as $x \to 0$ and assume $x_{in} \to 0$ as $n \to \infty$ for each i. Then there is a sequence k_n and a sequence $\hat{x}_{k_n}$ satisfying (ii) and (iii) and also (i'): $d(\hat{x}_{k_n}, x_{nk_n}) < \epsilon(\|x_{nk_n}\|^*)$. In terms of these conditions, theorems concerning the continuity of additive operations from Saks spaces into Banach or Fréchet spaces are derived. Other examples of "generalized" Saks spaces are obtained by considering Banach- or Fréchet-valued "measures" on an abstract space.

B. Gelbaum (Minneapolis, Minn.).

Schatten, Robert. "Closing-up" of sequence spaces. Amer. Math. Monthly 57, 603–616 (1950).

This is a discussion of two methods of completing a linear space of real number sequences $(u_1, u_2, \cdots)$, where only finitely many $u_i \neq 0$, with respect to a "symmetric" norm $\phi(u_1, u_2, \cdots)$ satisfying the three basic axioms for a norm plus the following conditions:

(a) $$\phi(u_1, u_2, \cdots) = \phi(\epsilon_1 u_{f(1)}, \epsilon_2 u_{f(2)}, \cdots),$$

where $\epsilon_i = \pm 1$, $f(n)$ is a one-to-one mapping of the integers on themselves; (b) $\phi(1, 0, 0, \cdots) = 1$. The first method (strong closure) consists in the construction of all sequences $(u_1, u_2, \cdots)$ for which $\lim_{n\to\infty} \phi(u_1, \cdots, u_n, 0, 0, \cdots) < \infty$; the second method is the Cantor-Méray completion of a metric space. The Cantor-Méray completion is always included in the strong closure and examples are given, which show the two can be distinct. For an n-dimensional space and a given ϕ one defines $\psi(v_1, \cdots, v_n)$ to be

$$\max_{|u_1|+\cdots+|u_n|=1} [u_1 v_1 + \cdots + u_n v_n] / \phi(u_1, \cdots, u_n),$$

the conjugate norm of ϕ. Then for this ψ, applied to the general linear spaces considered, one finds that the conjugate space of the Cantor-Méray completion is the set of sequences $(v_1, v_2, \cdots)$ for which $\lim_{n\to\infty} (v_1, \cdots, v_n, 0, 0, \cdots) < \infty$.

B. Gelbaum (Minneapolis, Minn.).

Prachar, Karl. Über bedingt konvergente Vektorreihen im Banach'schen Raum. Monatsh. Math. 54, 284–307 (1950).

The following problem is treated: Let E be a Banach space, $a_n \in E$, $|a_n| \to 0$, $\sum_1^\infty |a_n| = \infty$. Let $\mathfrak{M}$ be a set of orthogonal transformations of E. Then, for a given $a \in E$, can one choose transformations $M_i \in \mathfrak{M}$ so that $\sum_1^\infty M_i a_i = a$? Essential tools in the treatment are: (1) The functions $\mu(x) = \min_{\lambda \geq 0} |a - \lambda x|$, $\bar{\lambda}(x) = \sup \{\lambda : |a - \lambda x| = \mu(x)\}$ and $\underline{\lambda}(x) = \inf \{\lambda : |a - \lambda x| = \mu(x)\}$; (2) the cone (for $0 < \rho < 1$) $\mathfrak{K}\rho(a) = 0 \cup \{x : \mu(x) \leqq \rho |a|\}$. The set $\mathfrak{M}$ is said to have property E if for each a in the space there is a fixed ρ, $0 < \rho < 1$, such that for each b in the space there is an M in $\mathfrak{M}$ for which $Mb \in \mathfrak{K}\rho(a)$. It is shown that for sets $\mathfrak{M}$ having property E the problem is answered affirmatively. If $\mathfrak{M}$, in addition, is a finite set: $M_0, M_1, \cdots, M_l$, including the identity, and if we set $M_\lambda = M_r$ if $k \equiv r \pmod{l}$, then one can find M_{i_j} in $\mathfrak{M}$ so that $\sum_{j=1}^{\infty} M_{i_j} a_j = a$ and further so that $i_{j+1} = i_j$ or $i_{j+1} \equiv i_j + 1 \pmod{l}$. If, furthermore, $\mathfrak{M}$ is a finite group and if n_i is the number of elements of index not greater than n which are transformed by M_i, then, for a

suitable choice of the M_i, $\lim_{n\to\infty}(n_i/n)=1/l$. In particular, it is shown that if $\mathfrak{M}$ is the group of rotations of an n-simplex in n-dimensional Euclidean space (i.e., $\mathfrak{M}$ is the alternating group A_{n+1} on $n+1$ letters), then $\mathfrak{M}$ has property E. Further specializations relate the general problem to theorems of Rademacher [Math. Z. 11, 276–288 (1921)] and Auerbach [Studia Math. 2, 228–230 (1930)]. *B. Gelbaum.*

Damsteeg, Ira, and Halperin, Israel. The Steinitz-Gross theorem on sums of vectors. Trans. Roy. Soc. Canada. Sect. III. (3) **44**, 31–35 (1950).

Steinitz [J. Reine Angew. Math. **143**, 128–175 (1913)] showed that the Riemann derangement theorem for non-absolutely convergent series of real numbers generalizes in m-dimensional space: If $\sum_1^\infty v_n$ converges but not absolutely, where v_n are m-dimensional vectors, then its convergent rearrangements form a linear subspace. This was proved on the basis of the functions $K_m(c_1, c_2)$ defined as follows: Let $|u_1|\leqq c_1$, $|u_2|\leqq c_2$, $|v_i|\leqq 1$, $1\leqq i\leqq n$, $u_1+u_2+\sum_1^n v_i=0$, n arbitrary. There is a reordering of the v_i, v_i' for which $\sup_k|u_1+\sum_{i=1}^k v_i'|\leqq K_m(c_1, c_2)$, and $K_m(c_1, c_2)$ is the least number for which this is true for all n. By obtaining the estimates $K_m(0, 0)\geqq\frac{1}{2}(m+3)^{\frac{1}{2}}$, and for $m\geqq 2$, $K_m(1, 0)\geqq\frac{1}{2}(m+6)^{\frac{1}{2}}$, the authors show that the methods of Steinitz do not serve to extend his theorem to Hilbert space. *B. Gelbaum* (Minneapolis, Minn.).

Kantorovič, L. V. On Newton's method. Trudy Mat. Inst. Steklov. **28**, 104–144 (1949). (Russian)

Let X and Y be Banach spaces and $P(x)$ a (nonlinear) transformation from X to Y. A derivative (in the sense of Fréchet) $P'(x)$ may be defined and for fixed x is again a transformation from X to Y. The author wishes to investigate conditions under which the equation $P(x)=\theta$ may be solved by an analogue of Newton's method of successive approximations: $x_{n+1}=x_n-[P'(x_n)]^{-1}P(x_n)$. His main theorem is the following. Let the following conditions be fulfilled: (1) in the region defined by (*) below $P'(x)$ and $P''(x)$ exist and $\|P''(x)\|\leqq K$: (2) for a fixed value of the initial element x_0 the transformation $P'(x_0)$ has an inverse $\Gamma_0=[P'(x_0)]^{-1}$ with $\|\Gamma_0\|\leqq B_0$; (3) $\|\Gamma_0 P(x_0)\|\leqq\eta_0$; (4) $h_0=B_0\eta_0 K\leqq\frac{1}{2}$. Then there exists a solution x^* of $P(x)=\theta$ in the sphere about x_0 defined by (*) $\|x-x_0\|\leqq[1-(1-2h_0)^{\frac{1}{2}}]h_0^{-1}\eta_0$ and the rapidity of convergence of the sequence defined by Newton's method can be estimated from $\|x_n-x^*\|\leqq 2^{-n+1}(2h_0)^{2^n-1}\eta_0$. A slight modification will give uniqueness. Modifications of the recursion formula are also discussed, e.g.,

$$x_{n+1}=x_n-[P'(x_0)]^{-1}P(x_n).$$

Finally, applications are given to systems of algebraic equations, to nonlinear integral equations, and to the equation $Ax=\lambda x$ in Hilbert space, where A is self-adjoint. The paper is fairly self-contained, the necessary (elementary) parts of Banach space theory being developed in the paper. *J. V. Wehausen* (Providence, R. I.).

Mysovskih, I. P. On the convergence of Newton's method. Trudy Mat. Inst. Steklov. **28**, 145–147 (1949). (Russian)

The author proves a theorem similar to the one stated in the preceding review. The inverse of $P'(x)$ is assumed to exist and to be of uniformly bounded norm for all points x of a specified sphere about x_0, rather than just for x_0. This allows a larger bound for the quantity corresponding to h_0 and a very short proof. *J. V. Wehausen.*

Hornich, Hans. Zur Auflösung von Gleichungssystemen. Monatsh. Math. **54**, 130–134 (1950).

Let R be a finite dimensional vector space with elements $\mathfrak{x}$ and norm $|\mathfrak{x}|$, and let R' be a second such space with the same number of dimensions. Functions $\mathfrak{L}(\mathfrak{x})$, $\mathfrak{F}(\mathfrak{x})$ associate a point in R' to each point $\mathfrak{x}$ in a subset M of R. The author establishes the following theorem. Let $\mathfrak{F}(\mathfrak{x})$ be defined in $G=\{\mathfrak{x}: |\mathfrak{x}-\mathfrak{x}_0|\leqq\rho, \rho>0\}$, and assume that there exists a linear function $\mathfrak{L}(\mathfrak{x})$ with an inverse and a number q with $0\leqq q<1$ such that for every two points $\mathfrak{x}'$, $\mathfrak{x}''$ in G (*) $|\mathfrak{F}(\mathfrak{x}')-\mathfrak{F}(\mathfrak{x}'')-\mathfrak{L}(\mathfrak{x}'-\mathfrak{x}'')|\leqq q|\mathfrak{L}(\mathfrak{x}'-\mathfrak{x}'')|$; then to each point $\mathfrak{y}$ in R' with $|\mathfrak{y}-\mathfrak{F}(\mathfrak{x}_0)|\leqq\rho(1-q)K$, where $K=\min|\mathfrak{L}(\mathfrak{x}-\mathfrak{x}_0)|$ for all $|\mathfrak{x}-\mathfrak{x}_0|=1$, there exists one and only one point $\mathfrak{x}$ in G such that $\mathfrak{F}(\mathfrak{x})=\mathfrak{y}$. The proof, by an iteration procedure, shows that the solution of $\mathfrak{F}(\mathfrak{x})=\mathfrak{y}$ is $\lim_n \mathfrak{x}_n$, where $\mathfrak{x}_{n+1}=\mathfrak{x}_n+\mathfrak{L}^{-1}(\mathfrak{y}-\mathfrak{F}(\mathfrak{x}_n))$ for $n\geqq 0$. No assumption is made concerning the existence of derivatives. The paper contains an investigation of the class of functions $\mathfrak{L}(\mathfrak{x})$ which satisfy the condition (*) and results on the most rapidly converging sequences $\mathfrak{x}_n$. *G. B. Price.*

Katětov, Miroslav. Linear operators. I. Časopis Pěst. Mat. Fys. **75**, D9–D31 (1950). (Czech. English summary)

The author presents an introduction into the theory of Hilbert spaces. The titles of the chapters indicate the content: normed linear spaces; unitary spaces; complete unitary spaces; continuous linear mappings; linear functionals and weak convergence; linear operators. *František Wolf* (Berkeley, Calif.).

***Nakano, Hidegorô. Modern Spectral Theory.** Maruzen Co., Ltd., Tokyo, 1950. vi+323 pp. $3.00.

The principal aim of this book is the systematic presentation of various representation theories for lattice ordered linear spaces and rings. The author has brought together a great deal of material on this subject, much of which stems from his own research. To a certain extent, however, the treatment is oversystematized since practically the entire development is based on the notion of a projector; nowhere is the equally useful notion of a lattice ideal introduced. After an introductory chapter on point set topology, the author develops the background material of semi-ordered and lattice ordered linear spaces. For a given $p>0$ there corresponds to each $a>0$ of a continuous linear lattice R a third element $[p]a=V[a\wedge np|n=1, 2, \cdots]$. This correspondence can be extended over R, and defines a linear idempotent operator called a projector. With the usual notion of the resolution $[p_\lambda]$ of a projector $[p]$, $a_\lambda=[p_\lambda]a$ is called a resolution of a if $[p]a=a$. To each ordered pair $a, b\in R$ there exists a resolution a_λ of a such that $[a]b=\int_{-\infty}^{\infty}\lambda\, da_\lambda$. A second approach to the spectral theory considers as points the maximal collections of projectors having the finite intersection property. The space of such points is topologized in the usual way to be a locally compact Hausdorff space E; the "ratio" of pairs of elements in R are then represented as continuous numerically valued (infinity included) functions on E. The author then goes on to treat continuous semi-ordered normal ($a, b\geqq 0. \supset .ab\geqq 0$ and $[ab]=[a][b]$) rings. The pointwise representation of a^2/a gives a faithful representation of the lattice ring. A chapter is devoted to the continuous extensions of semi-ordered linear spaces and rings. The author then considers C-spaces of continuous functions on a compact Hausdorff space which vanish at "infinity." He shows that a ring or

lattice isomorphism between two such spaces induces a homeomorphism between the underlying spaces; he also obtains a characterization of such C-spaces. A final chapter deals with linear functionals on a semi-ordered lattice. The presentation is quite formal and contains no applications.

R. S. Phillips (Princeton, N. J.).

***Nakano, Hidegorô. Modulared Semi-Ordered Linear Spaces.** Maruzen Co., Ltd., Tokyo, 1950. i+288 pp.

In order to obtain a more detailed theory for universally continuous semi-ordered linear spaces R, the author has introduced the notion of a modular and investigated the properties of such modulared spaces. A functional $m(a)$ defined for $a \in R$ is called a modular if (1) $0 \leqq m(a) \leqq \infty$; (2) if $m(\xi a)=0$ for all $\xi \geqq 0$, then $a=0$; (3) to any $a \in R$ there exists an $\alpha>0$ such that $m(\alpha a)<\infty$; (4) $m(\xi a)$ is a convex function of $\xi \geqq 0$; (5) if $|a| \leqq |b|$, then $m(a) \leqq m(b)$; (6) if $a \wedge b=0$, then $m(a+b)=m(a)+m(b)$; and (7) if $0 \leqq a_\lambda \nearrow a$, then $m(a)=\sup m(a_\lambda)$. The first half of the book consists of preliminary material much of which is to be found in the book reviewed above. In the second half, the author first classifies modulars by a great variety of properties. For instance, a modular is said to be monotone complete if to any set $0 \leqq a_\lambda \nearrow$ such that $\sup m(a_\lambda)<\infty$ there exists an $a \in R$ for which $a_\lambda \nearrow a$. Another method of classifying modulars is by means of the modular function $\omega(\xi|a)=m(\xi \alpha a)$, where α is chosen so that $m(\xi a)>1$ for $\xi>\alpha$ and $m(\xi a) \leqq 1$ for $\xi \leqq \alpha$. If there exists a one-to-one correspondence between R and its conjugate space $\bar{R}$ $(a \to a^{\bar{R}})$ such that (1) $(-a)^{\bar{R}}=-a^{\bar{R}}$; (2) $a^{\bar{R}} \geqq b^{\bar{R}}$ if and only if $a \geqq b$; and (3) $a^{\bar{R}}(a)=0$ for $a \geqq 0$ implies $a=0$, then R is said to be conjugately similar. The author is able to characterize such spaces by means of the existence of a certain type of modular on R. In the final chapter, the author considers the problem of extending a modulared semi-ordered linear space to a universally continuous modulared semi-ordered linear space.

R. S. Phillips (Princeton, N. J.).

Tagamlitzki, Y. Sur quelques applications de la théorie générale des espaces vectoriels partiellement ordonnés. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **45**, 263–286 (1949). (Bulgarian. French summary)

This paper is concerned with partially ordered linear spaces over an ordered field (which in all applications mentioned is either the rational or real field). Let the order relation in such a space R be denoted by the symbol $\leqq$. The following axioms are assumed: (1) $a=b$ implies $a \leqq b$; (2) $a \leqq b$ and $b \leqq a$ imply $a=b$; (3) $a \leqq b$ and $c \leqq d$ imply $a+c \leqq b+d$; $a \leqq b$ and $\lambda \geqq 0$ imply $\lambda a \leqq \lambda b$. A partially ordered linear space R is said to be normal (completely normal) if every finite (arbitrary) set of elements admitting an upper bound admits a least upper bound. As the author notes, this is weaker than the requirement that R be lattice ordered or complete-lattice ordered. A positive nonzero element $p \in R$ is said to be simple if $0 \leqq a \leqq p$ implies that $a=\lambda p$ for some scalar λ. [Reviewer's note. The elementary positive definite functions of Gel'fand and Raĭkov [Rec. Math. [Mat. Sbornik] N.S. **13**(**55**), 301–316 (1943); these Rev. **6**, 147] provide an obvious example of such elements.] A number of simple facts, most of them well-known, are proved concerning relations between suprema, infima, and addition in normal and completely normal spaces; the introduction of order in a linear space with convergence is also discussed. More interesting is the following result. Let R be completely normal. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of simple elements, let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of nonnegative scalars, and let $\sum_{n=1}^{\infty}\beta_n p_n$ be an infinite series whose partial sums are representable as the difference of monotone increasing bounded sequences, so that $b=\sum_{n=1}^{\infty}\beta_n p_n$ exists. Then, if a is any element such that $-b \leqq a \leqq b$, there exist scalars α_n such that $|\alpha_n| \leqq \beta_n$ $(n=1, 2, \cdots)$ and $a=\sum_{n=1}^{\infty}\alpha_n p_n$. Two similar theorems are also proved.

These results are applied to the study of a particular partially ordered linear space R^*, whose elements consist of all infinitely differentiable functions defined on some interval (a, b). Sums and real multiples are defined as usual. A partial ordering in R^* is defined by means of a sequence of numbers $x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq \cdots$ in (a, b) with limit b. A function $f \in R^*$ is nonnegative if $(-1)^k f^{(k)}(x) \geqq 0$ for $a<x<x_k$ $(k=0, 1, 2, \cdots)$. It is proved that R^* is completely normal. It is also proved that the Gončarov polynomials P_n [Ann. Sci. École Norm. Sup. (3) **47**, 1–78 (1930)], multiplied by $(-1)^n$, are simple elements of R^*. Finally, it is proved that a function f in R^* can be expanded in an absolutely convergent series of simple elements p_n of R^* (which may or may not be the Gončarov polynomials) if and only if there exists an absolutely convergent series $\sum_{n=1}^{\infty}\beta_n p_n=g$, where the coefficients are nonnegative, such that $|f^{(k)}(x)| \leqq (-1)^k g^{(k)}(x)$ for $a<x<x_k$ and $k=1, 2, 3, \cdots$. Thus a result is obtained paralleling those of Gončarov [loc. cit.].

E. Hewitt (Seattle, Wash.).

Šreĭder, Yu. A. The structure of maximal ideals in rings of measures with convolution. Mat. Sbornik N.S. **27**(**69**), 297–318 (1950). (Russian)

Let G be a commutative topological group satisfying the second axiom of countability, and let $\mathfrak{R}_G$ be the set of all countably additive complex-valued measures defined on the family of Borel sets in G. Let sums and complex multiples of elements of $\mathfrak{R}_G$ be defined in the usual fashion. For any two measures σ and φ in $\mathfrak{R}_G$, let the convolution $\varphi * \sigma$ be defined by the formula (1) $\varphi * \sigma(E)=\int_G \varphi(E-t)\,d\sigma(t)$, for every Borel set E in G. It is proved that the integral in (1) is well defined and that $\varphi * \sigma$ is an element of $\mathfrak{R}_G$. With the norm $\|\varphi\|$ defined as the sum of the variations of the real and imaginary parts of φ, for all $\varphi \in \mathfrak{R}_G$, $\mathfrak{R}_G$ becomes a commutative Banach algebra over the complex numbers. In the following, we identify maximal ideals of $\mathfrak{R}_G$ with homomorphisms of $\mathfrak{R}_G$ onto the complex numbers, in the standard way.

The algebra $\mathfrak{R}_R$ (R the additive group of real numbers) was described by Gel'fand [Rec. Math. [Mat. Sbornik] N.S. **9**(**51**), 51–66 (1941); these Rev. **3**, 51]; it was pointed out that the Fourier-Stieltjes transform (2) $F(t)=\int_{-\infty}^{+\infty} e^{ixt}\,d\varphi(x)$ produces a class of maximal ideals in $\mathfrak{R}_R$. More maximal ideals were identified by Gel'fand, Raĭkov, and Šilov [Uspehi Matem. Nauk (N.S.) **1**, no. 2(12), 48–146 (1946); these Rev. **10**, 258], but the class of all maximal ideals was not identified.

In the paper under review the author supplies details of previously announced results dealing with $\mathfrak{R}_G$ [Doklady Akad. Nauk SSSR (N.S.) **63**, 359–361 (1948); these Rev. **10**, 309]; he also establishes a number of new results. It is first shown that the general bounded linear functional L on $\mathfrak{R}_G$ has the form (3) $L(\varphi)=\int_G f_\varphi(t)\,d\varphi(t)$, where f_φ (a generalized function, in the author's terminology) is a complex-valued function defined on $\mathfrak{R}_G \times G$ φ-measurable for every $\varphi \in \mathfrak{R}_G$ such that $\sup_\varphi [\operatorname{ess\,sup}_{t \in G} |f_\varphi(t)|]=\|L\|$, and such that if σ is absolutely continuous with respect to φ, then $f_\sigma(t)=f_\varphi(t)$ except for a set of $|\sigma|$-measure zero. The proof given is not complete, but the theorem is easily verified by

use of the Radon-Nikodym theorem. A generalized function f_φ is called a generalized character if $f_\varphi(s+t)=f_\varphi(s)f_\varphi(t)$ for all $\varphi\epsilon\mathfrak{R}_G$, the equality holding for all pairs (s, t) except for a set of $|\varphi|\times|\varphi|$ measure zero, and if the norm of f_φ as a linear functional is 1. The author proves that every functional (3) which is an algebra-homomorphism onto the complex numbers is produced by a generalized character; and he also states that if f_φ is a generalized character, then the functional (3) is an algebra-homomorphism of $\mathfrak{R}_G$ onto the complex numbers. However, a part of the proof is omitted which the reviewer is unable to supply.

The remainder of the paper deals with the algebra $\mathfrak{R}_R$. It is proved that maximal ideals in this algebra exist in addition to those found by Gel'fand, Raĭkov, and Šilov [loc. cit.]. The proof given for theorem 4, essential in this construction, appears to be incorrect; the theorem, however, is true. The author next shows that $\mathfrak{R}_R$ is nonsymmetric. For $\sigma\epsilon\mathfrak{R}_R$, let the measure $\tilde{\sigma}$ be defined by the relation $\tilde{\sigma}(E)=\overline{\sigma(-E)}$, for all Borel sets E. By the uniqueness theorem for Fourier-Stieltjes transforms, the only possible involution in $\mathfrak{R}_R$ making $\mathfrak{R}_R$ symmetric is $\sigma\to\tilde{\sigma}$. The author constructs a measure σ and a maximal ideal M such that $M(\sigma)=1$ and $M(\tilde{\sigma})=0$. This measure σ is used, with some facts concerning the topology of the space of all maximal ideals in $\mathfrak{R}_R$, to establish a theorem stated by Wiener and Pitt [Duke Math. J. 4, 420–436 (1938)]. This theorem asserts the existence of a Fourier-Stieltjes transform F which is bounded away from zero in absolute value and which has the property that F^{-1} is not a Fourier-Stieltjes transform. The proof offered by Wiener and Pitt being obscure, it is of interest that the measure $\sigma-\tilde{\sigma}-\epsilon_0$ has a Fourier-Stieltjes transform satisfying the conditions of Wiener and Pitt (ϵ_0 is the measure 1 concentrated at 0). The paper closes with another construction showing that the set of ordinary Fourier-Stieltjes transforms is not the boundary of the space of all maximal ideals in $\mathfrak{R}_R$. *E. Hewitt.*

Šnol', I. È. Closed ideals in the ring of continuously differentiable functions. Mat. Sbornik N.S. **27(69)**, 281–284 (1950). (Russian)

Let D_1^N be the class of functions which are defined and of class C^1 on the unit cube Ω in N-dimensional Euclidean space E_N. Under the ordinary algebraic operations and norm

$$\|f\|=\max_{P\epsilon\Omega}\left(|f(P)|+\left|\frac{\partial f(P)}{\partial x_1}\right|+\cdots+\left|\frac{\partial f(P)}{\partial x_N}\right|\right),$$

where $P=(x_1, \cdots, x_N)$, D_1^N is a normed ring. The author proves that every closed ideal in D_1^N is an intersection of closed primary ideals. A primary ideal is one which is contained in exactly one maximal ideal. It is shown that every closed primary ideal in D_1^N is equal to the set of all $f\epsilon D_1^N$ which vanish at some point $P_0\epsilon\Omega$ and whose total differentials $df=\sum(\partial f/\partial x_i)\,dx_i$ at P_0 vanish identically for $(dx_1, \cdots, dx_N)$ in some linear subspace of E_N determined by the ideal. *C. E. Rickart* (New Haven, Conn.).

Bourgin, D. G. Multiplicative transformations. Proc. Nat. Acad. Sci. U. S. A. **36**, 564–570 (1950).

Soit S un espace compact, $C(S)$ l'espace de Banach des fonctions numériques continues dans S. L'auteur détermine les fonctions numériques continues M sur $C(S)$ telles que $M(xy)=M(x)M(y)$. Par passage aux logarithmes, on voit que pour les fonctions x dont le minimum dans S est >0, on a $M(x)=\exp\int_S \log x(t)\,d\mu(t)$, où μ est une mesure sur S; l'auteur montre que μ est positive et a un support dénombrable et fermé (ce qui généralise la propriété analogue des caractères de $C(S)$, pour lesquels on sait que le support se réduit à un point); cette propriété se voit en montrant que dans le cas contraire une suite de fonctions x_n dont chacune est nulle dans un ensemble de mesure >0 pourrait tendre uniformément vers une fonction x pour laquelle $\log x(t)$ est intégrable. L'auteur étudie d'autre part les transformations T de $C(S_1)$ dans $C(S_2)$ qui sont multiplicatives; il montre que cette dernière propriété est entraînée par des conditions en apparence moins restrictives sur T [p. 565, ligne 6, la seconde parenthèse de la formule est mal placée, les parenthèses étant d'ailleurs inutiles dans cette formule; ligne 7, le premier des trois termes de l'inégalité est inutile]. *J. Dieudonné* (Baltimore, Md.).

Ramaswami, V. Normed algebras, isomorphism and the associative postulate. J. Indian Math. Soc. (N.S.) **14**, 47–64 (1950).

The Gelfand-Mazur theorem asserts that a complex Banach division algebra is necessarily the complex field. A number of generalizations are known in which weaker hypotheses imply the same conclusion. In the present paper the author has attempted to get the result with an absolute minimum of hypotheses. He even weakens the triangular inequality for the norm and the associative law for multiplication; the latter in several ways. In all he gives six different sets of sufficient conditions that a generalized complex Banach algebra be the complex field. The problem of deciding when a real Banach algebra is the reals, complexes, or quaternions is treated in the same spirit. *G. W. Mackey* (Cambridge, Mass.).

Godement, R. Sur la théorie des représentations unitaires. Ann. of Math. (2) **53**, 68–124 (1951).

Infinite analogues of classical decomposition theorems for representations of groups and algebras do not hold, except in rather special cases, unless the classical notion of direct sum is generalized to allow "continuous" decompositions into "infinitesimal parts." J. von Neumann has recently developed a theory of such a notion and used it to decompose weakly closed operator rings into factors [same Ann. (2) **50**, 401–485 (1949); these Rev. **10**, 548]. F. I. Mautner has made use of von Neumann's theory to decompose self-adjoint families of operators into irreducible parts [ibid. **51**, 1–25 (1950); these Rev. **11**, 324]. The bulk of the paper under review is devoted to an exposition of an alternative form of von Neumann's theory and to a proof in this context of Mautner's decomposition theorem.

Both von Neumann and the author deal with a space Y having a measure and an assignment to each point y in Y of a Hilbert space $\mathfrak{H}_y$. The elements of the continuous sum $\int\mathfrak{H}_y\,d\mu(y)$ are functions (or rather, equivalence classes of functions) from Y to the $\mathfrak{H}_y$ for which $\int\|f(y)\|^2 d\mu(y)<\infty$. Unfortunately, the family of all such functions need not be additive and the problem of defining $\int\mathfrak{H}_y\,d\mu(y)$ is that of choosing a suitable additive subfamily. For von Neumann Y is the real line and the additive family is defined axiomatically. There may be many families satisfying the axioms but all are equivalent in a reasonable sense. For the author Y is a general locally compact space and there is given in advance a family of functions which is to be part of $\int\mathfrak{H}_y\,d\mu(y)$. The assumed properties of this family are such that there is a unique von Neumann family containing it and the author's construction yields a unique object. The functions in the family have continuity properties with respect to the topology in Y and arise in a natural way in

applications. A feature of the present treatment is the substitution of topology for measure theory. The measure μ is given as a linear functional on the continuous functions with compact support rather than as a set function and the development is analogous to that of integration theory from the linear functional point of view. In fact, the author points out that in a certain sense the theory of integration may be looked upon as included in the theory of continuous direct sums.

Chapters I and II contain certain preliminary material on representations of self-adjoint algebras; particularly a treatment of the spectral theorem for commutative self-adjoint operator algebras and a generalization of the commutative Plancherel theorem. The theory of continuous direct sums is developed in chapter III and Mautner's theorem is proved in chapter IV. For Mautner's theorem certain denumerability restrictions are needed. There are numerous subsidiary results and an appendix dealing with the decomposition of invariant measures into ergodic parts.

G. W. Mackey (Cambridge, Mass.).

Barbašin, E. A. On homomorphisms of dynamical systems. Mat. Sbornik N.S. **27(69)**, 455–470 (1950). (Russian)

This paper contains proofs and several applications of theorems previously announced by the author [Doklady Akad. Nauk SSSR (N.S.) **61**, 429–432 (1948); these Rev. **10**, 49]. *W. H. Gottschalk* (Philadelphia, Pa.).

Gurevič, A. A., and Rohlin, V. A. Approximation theorems for measurable flows. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 537–548 (1950). (Russian)

Detailed proofs of results announced earlier [Doklady Akad. Nauk SSSR (N.S.) **64**, 619–620 (1949); these Rev. **10**, 549]. *P. R. Halmos* (Chicago, Ill.).

Calculus of Variations

***Fox, Charles. An Introduction to the Calculus of Variations.** Oxford University Press, New York, N. Y., 1950. viii+271 pp. $4.50.

This book, designed as a text for undergraduate students, includes a large number of examples, and devotes chapters 5, 6, and 7 to applications to mechanics, relativity, and elasticity. Unfortunately, the meaning of terms used is often very vaguely conveyed, and proofs are often very unsatisfactory. The errors and defects are too numerous to mention in detail. *L. M. Graves* (Chicago, Ill.).

Cesari, Lamberto. Problemi di calcolo delle variazioni e questioni connesse. Rivista Mat. Univ. Parma **1**, 293–304 (1950).

Lecture given at the 3d congress of the Unione Matematica Italiana, Pisa, September, 1948.

Magenes, Enrico. Un'osservazione sulle condizioni necessarie per la semicontinuità degli integrali di Fubini-Tonelli. Rend. Sem. Mat. Univ. Padova **19**, 44–53 (1950).

For the functional

$$I(y_1, y_2) = \int_a^b \int_c^d f(x, z, y_1(x), y_2(z), y_1'(x), y_2'(z))\, dz\, dx$$

the author shows that a necessary condition for lower semi-continuity is that there exist a function $L(x, z, y_1, y_2)$ such that $f(x, z, y_1, y_2, y_1', y_2') > -L(x, z, y_1, y_2)(1+y_1'^2)^{\frac{1}{2}}(1+y_2'^2)^{\frac{1}{2}}$. The proof is made under the supplementary hypothesis that the function f is continuous in (x, z, y_1, y_2) uniformly with respect to (y_1', y_2'). A similar theorem is stated for the functional

$$I(y) = \int_a^b \int_a^b f(x, z, y(x), y(z), y'(x), y'(z))\, dz\, dx.$$

L. M. Graves (Chicago, Ill.).

Magenes, Enrico. Sulle equazioni di Eulero relative ai problemi di calcolo delle variazioni degli integrali di Fubini-Tonelli. Rend. Sem. Mat. Univ. Padova **19**, 62–102 (1950).

The author gives sufficient conditions that a minimizing curve for the integral

$$I(y_1, y_2) = \int_a^b \int_c^d f(x, z, y_1(x), y_2(z), y_1'(x), y_2'(z))\, dz\, dx$$

should be an extremaloid, i.e., should satisfy

$$\int_a^x \int_c^d f_{y_1}(x, z, \cdots)\, dz\, dx - \frac{d}{dx}\int_a^x \int_c^d f_{y_1'}(x, z, \cdots)\, dz\, dx = c_1,$$

$$\int_c^z \int_a^b f_{y_2}(x, z, \cdots)\, dx\, dz - \frac{d}{dz}\int_c^z \int_a^b f_{y_2'}(x, z, \cdots)\, dx\, dz = c_2.$$

In particular, these equations are satisfied by a Lipschitzian minimizing curve. In case the minimizing curve is of class C', the equations may be differentiated once with respect to x and z, respectively, to obtain the Euler equations. It is shown by an example that minimizing extremals joining neighboring points do not necessarily lie near these points. Theorems are given on the existence of extremals joining two given points, or satisfying given initial conditions, and on the uniqueness of extremals. These depend heavily, both in statement and proof, on the author's previous memoirs [Ann. Scuola Norm. Super. Pisa (3) **2** (1948), 1–38 (1950); **3** (1949), 95–131 (1950); these Rev. **12**, 267]. The paper concludes with a brief discussion of

$$I(y) = \int_a^b \int_a^b f(x, z, y(x), y(z), y'(x), y'(z))\, dz\, dx.$$

Attention should be called to a detailed discussion of $I(y)$, including sufficient conditions for a semistrong relative minimum, and examples showing that the family of extremals may depend effectively on two, three, or four parameters; this appears in the work of A. R. Jacoby [Thesis, University of Chicago, 1946; obtainable on microfilm from the University of Chicago Library].

L. M. Graves (Chicago, Ill.).

Gillis, Paul P. Intégrales doubles du calcul des variations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 403–412 (1950).

The author considers a double integral

$$I(z) = \int\int_D (q^2 r - 2pqs + p^2 t + 6f(x, y, p, q))\, dx\, dy,$$

where p, q, r, s, t denote first and second partial derivatives of $z(x, y)$. It is readily seen that the boundary values prescribed for $z(x, y)$ may be required to be zero, without changing the form of the problem. The Euler equation $rt - s^2 + f_{pp}r + 2f_{pq}s + f_{qq}t + f_{pz} + f_{qy} = 0$ is derived. When

the function f is a polynomial in p and q of at most the second degree, with coefficients functions of x and y, the author shows that an extremal lying in the class K^+ of functions $z(x, y)$ vanishing on the boundary and such that $(f_{pp}+t)(f_{qq}+r)-(f_{pq}-s)^2>0$, $f_{pp}+t>0$, furnishes a proper absolute minimum for $I(z)$ in this class K^+. An extremal lying in the class K^- (in whose definition the condition $f_{pp}+t<0$ replaces $f_{pp}+t>0$) furnishes a proper absolute maximum for $I(z)$ in K^-. Since these classes do not overlap, there is no indication from these results that the minimum value is less than the maximum value. In the final paragraph a generalization of the restriction that f be a polynomial is indicated. *L. M. Graves* (Chicago, Ill.).

Morse, Marston, and Transue, William. A characterization of the bilinear sums associated with the classical second variation. Ann. Mat. Pura Appl. (4) **28**, 25–68 (1949).

The second variation of a classical variational integral on $[0, 1]$ is the quadratic form $\hat{J}$ associated with a bilinear form

$$J(u, v; P, Q, R, E)=\int_0^1 [R(s)\dot{u}(s)\dot{v}(s) + Q(s)\{u(s)\dot{v}(s)+v(s)\dot{u}(s)\}+P(s)u(s)v(s)]\,ds+E(u, v),$$

where E depends only on the values of u and v at 0 and 1; here u and v are in the space C of functions continuous on $[0, 1]$, and $\dot{u}$ and $\dot{v}$ are in L_2. We define $|\eta|_A=\sup|\eta(s)|$ if A is C, even if η is not continuous; $|\eta|_A$ is the usual L_2 norm if A is L_2. Let k be defined on the Cartesian product $E'\times E''$ of intervals of real numbers. Let A and B be C on L_2. We assume k to have certain left-continuity properties which we shall not mention further. Partition E', E'' into intervals δ_r', δ_n''; let η have constant value η_r on δ_r', let ζ have constant value ζ_n on δ_n'', and let $\Delta_{rn}(k)$ be the mixed second difference of k over $\delta_r'\times\delta_n''$. The supremum of $\sum_{r,n}\eta_r\zeta_n\Delta_{rn}(k)$ for all such partitions and all such η, ζ for which $|\eta|_A\leqq 1$ and $|\zeta|_B\leqq 1$ is called the A-B-variation of k over $E'\times E''$, and denoted by $h(A, B, k)$. Let E', E'' be $[0, 1]$, and A and B either C or L_2. The most general bilinear functional over $A\times B$ is given for $x\in A$ and $y\in B$ by $F(x, y; k)=\int_0^1 x(s)\,d_s\int_0^1 y(t)\,d_t k(s, t)$, k having finite variation $h(A, B, k)$. Thus the second variation form J is a special case of $S(u, v; p, q, r)=F(\dot{u}, \dot{v}; r)+F(u, \dot{v}; q)+F(v, \dot{u}; q)+F(u, v; p)$ for u, v integrals of functions in L_2. Both for $\hat{J}$ and for the quadratic form $\hat{S}$ associated with S there is a natural definition of first variation and of extremal. The object of this paper is to characterize the forms $\hat{J}$ among the larger class of forms $\hat{S}$ by properties of the extremals. Subject to the mild requirement that the form can be transformed into one with $q=0$, the characterizing properties are: (α) Each extremal u of S on $[0, 1]$, when regarded as on $[a, b]\subset[0, 1]$ defines an extremal of the correspondingly contracted form $\hat{S}_a^b$; (β) corresponding to each $c\in[0, 1]$ there is an extremal u_c such that $u_c(c)=1\cdot(g)$. There are extremals u, v of $\hat{S}$ whose Wronskian is almost everywhere $\neq 0$, while not every u with $u\in L_2$ is an extremal of $\hat{S}$. *E. J. McShane.*

Nickel, K. Lösung eines speziellen Minimumproblems. Math. Z. **53**, 21–52 (1950).

Let $S(x, y)=\pi^{-1}\log[\sin\frac{1}{2}|x+y|/\sin\frac{1}{2}|x-y|]$. The author wishes to minimize the integral $\int_0^\pi\int_0^\pi f(x)f(y)S(x, y)\,dx\,dy$ for $f\in L_p$ subject to the side conditions

$$(1)\qquad \int_0^\pi\int_0^\pi f(x)h_n(y)S(x, y)\,dx\,dy=A_n,$$

$n=1, \cdots, N$, and

$$(2)\qquad \int_0^\pi\int_0^\pi f(x)f(y)g_m(y)S(x, y)\,dx\,dy=B_m,$$

$m=1, \cdots, M$, where the h_n and g_m are fixed functions and the A_n and B_m fixed constants. In a somewhat simpler formulation this problem occurs in airfoil theory. The problem is formulated in a linear space R having an inner product (f, g) with the usual properties. In addition, there are postulated two types of multiplication between elements f of R and elements h of $R_0\subset R$ such that $f*h\in R$, $f\overset{*}{*}h\in R$, and $(f, g*h)=(f\overset{*}{*}h, g)=(f, g, h)$; moreover, a unit $e\in R_0$ is postulated with $f*e=f\overset{*}{*}e=f$ and also a finite distributivity property. The minimum problem now becomes that of minimizing (f, f) subject to $(f, h_n)=A_n$ and $(f, f, g_m)=B_m$. A theorem, too long to state here, is proved concerning conditions for the existence of this minimum. This is then applied to the original problem and it is shown that, provided the B_m are properly restricted, there is a solution which may be obtained from a Fredholm integral equation of the second kind. The statement of all the conditions (corresponding to the choice of R_0, etc.) is too elaborate for a brief statement. *J. V. Wehausen* (Providence, R. I.).

Theory of Probability

***Fréchet, Maurice. Généralités sur les probabilités. Éléments aléatoires. 2d ed. Traité du calcul des probabilités et de ses applications, tome I, fasc. 3, premier livre.** Gauthier-Villars, Paris, 1950. xvi+355 pp.

The second edition of this useful book is the same as the first [1937] except for scattered minor changes and the addition of a short chapter on abstract-valued random variables. *J. L. Doob* (Urbana, Ill.).

***Fortet, Robert. Calcul des probabilités. Centre d'études mathématiques en vue des applications. A. Application des théories mathématiques, Vol. 1.** Centre National de la Recherche Scientifique, Paris, 1950. 330 pp. 1,200 francs.

The author states in his preface: This book was written for the use of physicists, engineers, biologists, statisticians, indeed in general for all those, more or less numerous, who have to use the calculus of probability, sometimes in its most advanced parts, although they are not experts in probability or even mathematicians by profession. Existing books in French, in the opinion of the author, are either too elementary or intended for specialists in probability. The author was invited to write the book by the section on probability of the Center for Applied Mathematics of France, and discussed his book during its preparation with its other members, who comprise three physicists, a biologist, a mathematician (Fréchet), and a chemist. A preface by the chairman, F. Joliot-Curie, describes the origin of the book.

The book begins with first principles and has an enormous scope. Proofs are largely omitted, but the book abounds in heuristic explanations designed to make the theorems intelligible and meaningful. Logical development is stressed. Numerous examples enhance the reader's understanding. The extent of the book may be judged by some of the topics discussed: the strong law of large numbers, law of the iterated logarithm, asymptotic expansions of convolutions. There is a long chapter on Markoff chains, and another on

statistics. The last chapter of about forty pages discusses eight widely different and interesting applications of probability to various problems. *J. Wolfowitz.*

***Feller, William. An Introduction to Probability Theory and Its Applications.** Vol. I. John Wiley & Sons, Inc., New York, N. Y., 1950. xii+419 pp. $6.00.

Afin d'éviter toute notion mathématique élevée (théorie de la mesure, etc.) et pour que l'ouvrage soit utilisable par des étudiants débutants, l'auteur s'est limité aux questions où n'interviennent qu'un nombre dénombrable d'éventualités; mais à propos de ces questions simples il aborde les problèmes les plus avancés de calcul des probabilités, dont beaucoup n'avaient pas encore été exposés dans un livre, de sorte que l'ouvrage est également du plus haut interêt pour les spécialistes. Le texte inclut un très grand nombre d'exemples ou d'exercises proposés, généralement originaux et remarquablement choisis.

Dans l'introduction et au chapitre 1, l'auteur présente les notions d'épreuve, d'évènement et de probabilité; celle-ci est décrite axiomatiquement, les points de vue concrets, en particulier la théorie de von Mises, n'étant signalés que comme interprétation (de sorte que l'ensemble du calcul des probabilités sera fondé comme théorie purement mathématique et autonome); le chapitre 2 est consacré à l'analyse combinatoire et à la formule de Stirling, dont les chapitres 3 et 4 décrivent des applications classiques (jeu de pile ou face dans le cas de Bernoulli) ou beaucoup moins communes (statistiques de Bose-Einstein et Fermi-Dirac, problèmes d'occupation, et de temps de retour). Le chapitre 5 est consacré à la composition et à la répétition des épreuves, la probabilité conditionnelle et l'indépendance stochastique (avec la théorie chromosomique comme illustration). Les chapitres 6 et 7 étudient le jeu de pile ou face (cas de Bernoulli et de Poisson); la loi de Poisson, et celle de Laplace sont obtenues approximations de la distribution de Bernoulli (ces problèmes d'approximations, bien illustrés de valeurs numériques, sont traités plus complètement et plus simplement qu'il n'est habituel); la loi des grands nombres (cas de Bernoulli) est obtenue comme conséquence de l'approximation Laplacienne; enfin, toujours dans le cas de Bernoulli, au chapitre 8, l'auteur établit la loi forte des grands nombres et la loi du logarithme itéré (énoncé de Khintchine). Les chapitres 9 et 10 sont consacrés aux variables aléatoires et à leurs espérances mathématiques (l'auteur se limitant dans l'exposition aux variables aléatoires à nombre fini de valeurs possibles; les inégalités de Chebyschev et de Kolmogoroff sont établies), puis à l'addition des variables aléatoires indépendantes: la loi des grands nombres dans le cas de variables aléatoires de même distribution (sans supposer l'existence de la variance); la loi forte des grands nombres dans le cas de distributions quelconques (et en particulier le théorème de Kolmogoroff pour des distributions identiques) sont établies. Au chapitre 11, l'auteur considère des variables aléatoires à valeurs entières et introduit leurs fonctions génératrices (applications aux distributions binomiales de Pascal; aux problèmes de temps d'attente et de réactions en chaine); les chapitres 12, 13, 14 exposent la théorie de l'auteur sur les "événements recurrents," et ses applications: loi de l'arc sinus; problèmes de renouvellements; dans le jeu de pile ou face (cas de Bernoulli); temps de retour; ruines des joueurs; promenade au hasard; et probabilités d'absorption. De la promenade au hasard dans le cas de Bernoulli, l'auteur passe (passages à la limite et extensions diverses) aux processus permanents continus, aux promenades au hasard à plusieurs dimensions, aux tests sequentiels. Au chapitre 15, l'auteur introduit les chaines de Markoff (cas fini ou dénombrable et homogène discret) et les étudie très simplement par sa méthode des événements recurrents (qui se confond à peu près dans ce cas avec la méthode suivie par Kolmogoroff pour le cas dénombrable; signalons à ce propos que le cas dénombrable a été considéré pour la première fois non par Kolmogoroff comme il est dit (p. 343), mais par Hostinsky, puis le rapporteur [C. R. Acad. Sci. Paris **201**, 184–186 (1935)]); l'étude est reprise au chapitre 16 par la méthode algébrique classique (cas fini); des cas finis ou denombrables homogènes permanents sont étudiés au chapitre 17 (processus de Poisson, exemples empruntés à la biologie ou la téléphonie), où les équations de Kolmogoroff sont obtenues (même pour une chaine non homogène). *R. Fortet* (Caen).

Richter, Hans. Zur Begründung des Inklusions- und Repräsentationsschlusses der mathematischen Statistik. Mitteilungsblatt Math. Statist. **2**, 83–89 (1950).

Attempt at validating Bayes' formula in the case of unknown a priori probabilities by assigning to it the meaning of a subjective probability. *G. E. Noether.*

Dufresne, Pierre. Problèmes de dépouillements. I. Problèmes intéressant deux candidats. Gaz. Mat., Lisboa **11**, nos. 44–45, 8–14 (1950).

This is another treatment of the classical problem of ballots. If candidate A [B] has a [b] votes, the probability is evaluated that in the counting A will always be ahead of B, that A will never be behind by more than m, and B never be behind by more than n votes, and so on. *J. L. Doob* (Urbana, Ill.).

Kakehashi, Tetsujiro. On the density of Lissajous' figures. J. Osaka Inst. Sci. Tech. Part I. **1**, 87–88 (1949).

Evaluation of some elementary asymptotic distributions. *J. L. Doob* (Urbana, Ill.).

Münzner, Hans. Zur Abschätzung von Wahrscheinlichkeiten. Mitteilungsblatt Math. Statist. **2**, 130–137 (1950).

Practical sharpening of the Chebyshev inequality in the direction of Gauss, assuming that the density function $g(s)$ is nonincreasing as $|s-a|$ increases and convex towards the s-axis if $|s-a| \geqq$ some constant. The author does not mention similar work of Fréchet [Généralités sur les probabilités. Variables aléatoires. Traité du calcul des probabilités et de ses applications, tome I, fasc. 3, livre 1, Gauthier-Villars, Paris, 1937, chapter 4]. *K. L. Chung.*

Nabeya, Seiji. On a relation between exponential law and Poisson's law. Ann. Inst. Statist. Math., Tokyo **2**, 13–16 (1950).

If $\{X_n\}$ is a sequence of nonnegative mutually independent random variables with a common distribution function F, and if $N_x = \max n$ with n subject to the condition $\sum_1^n X_j \leqq x$, and if N_x has the Poisson distribution for every x, it is proved that F is given by the density ce^{-cx} $(x>0)$ for some $c>0$. *J. L. Doob* (Urbana, Ill.).

Koopman, B. O. Necessary and sufficient conditions for Poisson's distribution. Proc. Amer. Math. Soc. **1**, 813–823 (1950).

The nth set consists of n independent trials $(n=1, 2, \cdots)$; p_{nk}, $k=1, \cdots, n$, is the probability that the kth trial in

the nth set shall succeed; $P_n(s)$ is that of exactly s successes in the nth set; $P(s)=e^{-m}m^s/s!$. It is shown that a necessary and sufficient condition for $\lim_{n\to\infty} P_n(s)=P(s)$ is (1) $\lim_{n\to\infty}\sum_{k=1}^{n}p_{nk}=m$; (2) $\lim_{n\to\infty}\max_{1\le k\le n}p_{nk}=0$. The more general problem of necessary and sufficient conditions for the distribution of the sum of independent random variables $X_{n1}+\cdots+X_{nn}$ to approach a given infinitely divisible law (including Poisson's) was solved by Gnedenko [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 1939, 181–232, 643–647; these Rev. 1, 341] and Doeblin [Bull. Sci. Math. (2) 63, 23–32, 35–64 (1939)], under the preliminary assumption that $\lim_{n\to\infty}\max_{1\le k\le n}\Pr(|X_{n,k}|>\epsilon)=0$ for every $\epsilon>0$. In the author's case this last condition reduces to (2) but is derived by him as necessary. Special cases and the speed of approach are discussed.

K. L. Chung (New York, N. Y.).

Paulson, Edward. Some limiting distributions related to the sum of a random number of random variables. Proc. Amer. Math. Soc. 1, 625–629 (1950).

Let N be an integer-valued random variable with a distribution depending on a parameter λ, i.e., with Prob $(N=k)=\omega_k(\lambda)=\omega_k$, for $k=0, 1, \cdots$, and let $\alpha=\alpha(\lambda)=E(N)$, $\gamma^2=\gamma^2(\lambda)=E[(N-\alpha)^2]$. A statistic Y_1 is considered, with distribution function $F(u)=\text{Prob}\,(Y_1\le u)=\sum_{k=0}^{\infty}\omega_k F(u\mid N=k)$, and the existence of constants a and $c>0$ is assumed such that $\lim_{\lambda\to\infty}\int_{-\infty}^{+\infty}\exp[it(u-ak)/ck^{\frac{1}{2}}]\,dF(u\mid N=k)=g(t)$ is a characteristic function. Setting $\sigma^2=\alpha c^2+\gamma^2a^2$, $Z_1=(Y_1-a\alpha)/\sigma$, the author shows under various conditions on the limit behavior of α and α/γ^2 that, for $\lambda\to\infty$, the random variable Z_1 has a limiting distribution, and derives an explicit formula for the characteristic function of this distribution. This generalizes a result of Robbins [Bull. Amer. Math. Soc. 54, 1151–1161 (1948); these Rev. 10, 385] obtained for the case $Y_1=\sum_{i=1}^{N}X_i$, where the X_i are independent determinations of the same random variable. A result of a similar nature is obtained for a statistic Y_2 which is a generalization of the mean $N^{-1}\sum_{i=1}^{N}X_i$ of a random number of independent repetitions of a random variable X. *Z. W. Birnbaum.*

Rao, K. S., and Kendall, David G. On the generalized second limit-theorem in the calculus of probabilities. Biometrika 37, 224–230 (1950).

The proof of the Fréchet-Shohat theorem is modified in order to avoid the notion of uniformly convergent improper integrals. The continuity theorem and expansions of characteristic functions are utilized instead. The lemmas used are classical. The purpose is to "place a simple and attractive proof within the reach of every mathematical undergraduate." The reviewer, while finding the approach of interest, cannot see how the modifications answer the stated purpose. *M. Loève* (Berkeley, Calif.).

Prohorov, Yu. V. On the strong law of large numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 523–536 (1950). (Russian)

Very detailed proofs of previously announced theorems [Doklady Akad. Nauk SSSR (N.S.) 69, 607–610 (1949); these Rev. 11, 375]. Kolmogorov's necessary and sufficient conditions for the weak law of large numbers are used at an intermediate stage. In (6.4) read $\sum_{k=1}^{n}$ for $\sum_{k=1}^{\infty}$. The reviewer wishes to correct an inaccuracy in his review cited above: Kawata's condition (b) is not necessary but the modified (b′) is. *K. L. Chung* (New York, N. Y.).

Loève, Michel. On sets of probability laws and their limit elements. Univ. California Publ. Statist. 1, 53–87 (1950).

An abstract of this paper appeared some time ago [Proc. Nat. Acad. Sci. U. S. A. 35, 328–332 (1949); these Rev. 11, 188]. It remains to point out that several theorems concerning functions of uniformly bounded variation (essentially the difference of two distribution functions) possess some independent interest. Thanks to them and the redeployment of classical tools a group of theorems pertaining to the infinitely divisible laws due to Lévy, Khintchine, Gnedenko, and Doeblin are proved with less ado than before, and sometimes as consequences of more general theorems for dependent variables. Extensions to random vectors and to a random number of random variables are sketched. Misprints: a few integral signs are missing in 2°, p. 61; an E is missing in (103). *K. L. Chung* (New York, N. Y.).

Hostinský, Bohuslav. Revue des travaux publiés en 1935–1948 sur les chaines de Markoff et problèmes voisins. Časopis Pěst. Mat. Fys. 74 (1949), 48–62 (1950). (Czech. French summary)

Bochner, S. Partial ordering in theory of stochastic processes. Proc. Nat. Acad. Sci. U. S. A. 36, 439–443 (1950).

The usual method of defining a probability measure on the space of numerically valued functions, with domain an abstract space T, is to assign measures to sets of functions determined by conditions on the functions at finitely many T points, and then extend the measure to more general function sets. If α is a finite T set, this means that a finite-dimensional distribution is assigned to α, and that the finite-dimensional distributions satisfy certain consistency conditions. The α's can be made into a directed set, in terms of the inclusion relation, and then the procedure can be described as making correspond to each α in a directed set a measure in a finite-dimensional space R_α. The measures and R_α's satisfy certain consistency conditions. The author generalizes this approach, allowing the α's to range through any directed set and the R_α's to be Hausdorff spaces. He then specializes to continuous parameter Markov processes with stationary transition probabilities, and using a transformation of a type discussed in a previous paper [Proc. Nat. Acad. Sci. U. S. A. 35, 368–370 (1949); these Rev. 10, 720] he obtains a generalization of a Markov process in which the parameter space is k-dimensional space. The Fokker-Planck equations are correspondingly generalized.

J. L. Doob (Urbana, Ill.).

Itô, Kiyosi. Stochastic differential equations in a differentiable manifold. Nagoya Math. J. 1, 35–47 (1950).

The author solves the r-dimensional stochastic differential equation

$$dx^i(t,\omega)=a^i[t,x(t,\omega)]\,dt+\sum_{j=1}^{r}b_j^i[t,x(t,\omega)]\,dy^j(t,\omega),\qquad i=1,\cdots,r,$$

where the family of r-dimensional vector random variables $\{y^1(t,\cdot),\cdots,y^r(t,\cdot)\}$ defines an r-dimensional Brownian motion for t in a finite interval. For the one-dimensional case see his earlier paper [Proc. Imp. Acad. Sci. Tokyo 22, nos. 1–4, 32–35 (1946); these Rev. 12, 191]. He then generalizes to the corresponding system of differential equations on a differentiable manifold, expressed in terms of local parameters. The solution, under appropriate initial conditions and restrictions on the coefficients a^i and b^i, defines

a Markov process, and a generalized version of the Fokker-Planck equations is derived. *J. L. Doob* (Urbana, Ill.).

Bruno, Ghidoli, Gian Là legge sinusoidale delle serie derivate con somme mobili. Statistica, Milano **10**, 351–364 (1950).

MacDonald, D. K. C. Spontaneous fluctuations. Reports on Progress in Physics **12**, 56–81 (1949).
Expository (nonmathematical) paper.
J. L. Doob (Urbana, Ill.).

Domb, C. The statistics of correlated events. I. Philos. Mag. (7) **41**, 969–982 (1950).
Consider events occurring in such a way that the distances between r consecutive events are mutually dependent, but that the waiting times of the events which are further apart are statistically independent. For $r=1$ we get the familiar counter problem. Let $g(x_1, \cdots, x_r)$ be the joint density function of the intervals between r consecutive events. The author evaluates the moments of the distribution of the distance between events number x and $x+n$ in the steady state. He derives a formal expression for the expected effect at time t if each event emits an impulse which after time t has magnitude $f(t)$. *W. Feller* (Princeton, N. J.).

***Insolera, Filadelfo. Teorica dell'ammortamento.** Giulio Einaudi, Torino, 1950. 303 pp. 2400 Lire.
This is the third and final volume of the author's treatise on actuarial science, the first is "Teorica della sopravivenza" [Giappichelli, Torino, 1947], the second is "Teorica della capitalizzazioni" [Einaudi, Torino, 1949]. The present volume is divided into two parts (part I: "An outline of a general theory of amortization," part II: "On actuarial amortization and related problems in insurance").

In the first chapter of part 1 financial operations are discussed in general terms; the operations which are the subject of this volume are characterized by their extended duration and by the fact that the obligations of at least one party are payable in installments. In the second chapter of part 1 annuities are discussed in rather general terms, various modes of payment (i.e., payments constant, or in arithmetic, or in geometric progression) as well as different assumptions concerning the law of capitalization are considered. The usual formulae for annuities certain are derived and an approximation formula for the rate of interest of an annuity of known value is given. Chapter three of part 1 deals with various methods of amortization in general as well as in more specific terms. The subject of chapter four of part 1 is the amortization of obligations redeemable by lotteries. This introduces an element of randomness and simple probabilistic methods into the theory of interest. After giving the necessary definitions the theory of these obligations is studied in great detail. Chapter five of part 1 is devoted to the study of mathematical reserves and to the normal theory of risk connected with credit operations. Throughout the book the author uses the term normal theory of risk to emphasize the fundamental assumption that the random variables under investigation (i.e., profit or loss in a game of chance) are normally distributed. A brief survey of the theory of reserves in financial operations is given. The consideration of obligations redeemable by lotteries shows the analogy with the methods usually applied to life insurance. The average risk and the mean risk are introduced and some of the usual formulae are derived. The successive linear risks are studied in detail. A few applications to obligations redeemable by lotteries conclude this chapter. The first chapter of part 2 deals with annuities in life insurance. Constant as well as varying annuities are studied; continuous, yearly as well as fractional payment is considered. A rather superficial treatment of invalidity and a brief survey of compound survivorship annuities is included. The value of an annuity is computed assuming various laws of mortality. The second chapter of part 2 gives a brief but adequate discussion of periodic payments. The third chapter of part 2 deals in a conventional manner with the theory of premium reserves. The ideas of Zillmer, Hoeckner, and Engelbrecht are presented to familiarize the reader with the methods of accounting for the expenses. Many other topics such as the recurrence formula and Thiele's differential equation are also included. The fourth chapter of part 2 contains general remarks on insurance and its economical and legal aspects. This leads to certain technical problems which are mostly connected with the preparation of the technical balance. The sources of profit are analyzed but methods for the distribution of the surplus are not discussed, however some pertinent references are given. The fifth and last chapter of part 2 is entitled "On Some Theories of Risk." The shortcomings of the normal theory of risk are recognized. Therefore the author mentions briefly alternate theories such as Medolaghi's theory of hypernormal dispersion and Lundberg's collective theory of risk. No details are given, even the presentation of the normal theory of risk is too sketchy. For instance, Hattendorff's theorem or Hausdorff's formulae are not even mentioned nor the more recent results of A. Berger [Mathematik der Lebensversicherung, Springer, Wien, 1939]. The discussion of the collective theory of risk is even more cursory. The connection with the theory of random processes is not mentioned. Numerous references to the older literature supplement this chapter and will be very helpful to the reader who wishes to penetrate into the subject. It is only to be regretted that some of the more recent contributions have not been included.

The book as a whole reflects strongly the author's personal preferences and interests. Superficially this is already shown by the fact that 83 references are given to the author's own work, while very few other authors are mentioned more than 6 times. This is not intended to be a criticism but should help to explain why so much emphasis is placed on certain topics while others are treated very briefly. The book is pervaded by the spirit of presenting everything in the greatest possible generality. Thus great emphasis is placed on the general theory of capitalization; instead of a fixed rate of interest a function of capitalization is introduced at an early stage and a rather general mathematics of finance is constructed. While this general theory is presented in great detail some important parts of actuarial mathematics are treated very briefly. Some examples were already given above; in addition, it should be remarked that select tables are only briefly mentioned and that no reference is made to the many important applications of Makeham's formula. The reviewer regrets most the extremely brief treatment of invalidity. No theory of tables with multiple decrement and their application in actuarial mathematics is given, the fundamental paper by A. Loewy [S.-B. Heidelberger Akad. Wiss. Abt. A, no. 6 (1917)] is not mentioned in the list of references. However, a compensatory feature of the book is the detailed presentation of the work of the Italian actuaries; of this work, the book represents the first unified account. *E. Lukacs* (Washington, D. C.).

Mathematical Statistics

***Fisher, R. A. Contributions to Mathematical Statistics.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. Unpaged (1 plate). $7.50.

This volume contains reproductions by a photographic method of 42 papers written by the author, both alone and in collaboration with others. The papers were selected by the author, and he has prepared introductory remarks for each; corrections of errors have been written into the text. There are also introductory remarks to a 43rd paper (no. 1 in the sequence) which is not reproduced. The papers are the following, including the one not reproduced: (1) Metron 1, no. 4, 3–32 (1921); (2) Monthly Not. Roy. Astr. Soc. **80**, 758–770 (1920); (3) J. Agricultural Sci. **11**, 107–135 (1920); (4) Ann. Appl. Biol. **9**, 325–359 (1922); (5) J. Roy. Statist. Soc. **85**, 87–94 (1922); (6) ibid. **85**, 597–612 (1922); (7) Economica **3**, 139–147 (1923); (8) J. Roy. Statist. Soc. **87**, 442–450 (1924); (9) Atti del Congresso Internazionale dei Mathematici, Bologna, 1928, v. 6, pp. 95–100; (10) Philos. Trans. Roy. Soc. London. Ser. A. **222**, 309–368 (1922); (11) Proc. Cambridge Philos. Soc. **22**, 700–725 (1925); (12) Proc. International Mathematical Congress, Toronto, 1924, v. 2, pp. 805–813; (13) Econometrica **3**, 353–365 (1935); (14) Proc. Roy. Soc. London. Ser. A. **121**, 654–673 (1928); (15) Proc. Cambridge Philos. Soc. **24**, 180–190 (1928); (16) Proc. Roy. Soc. London. Ser. A. **125**, 54–59 (1929); (17) J. Ministry of Agriculture **33**, 503–513 (1926); (18) J. Agricultural Sci. **19**, 201–213 (1929); (19) Proc. Roy. Soc. Edinburgh **50**, 205–219 (1930); (20) Proc. London Math. Soc. (2) **30**, 199–238 (1929); (21) Proc. Roy. Soc. London. Ser. A. **130**, 16–28 (1930); (22) Proc. Cambridge Philos. Soc. **26**, 528–535 (1930); (23) Mathematical Tables, vol. 1, British Association for the Advancement of Science, 1931, pp. xxvi–xxxv; (24) Proc. Roy. Soc. London. Ser. A. **144**, 285–307 (1934); (25) Ann. Eugenics **6**, 391–398 (1935); (26) J. Roy. Statist. Soc. **98**, 39–54 (1935) [original paper extended to p. 82]; (27) Proc. Amer. Acad. Arts Sci. **71**, 245–258 (1936); (28) Ann. Eugenics **7**, 189–193 (1936); (29) ibid. **7**, 303–318 (1937); (30) Revue de l'Institut International de Statistique **5**, 307–320 (1937); (31) Ann. Eugenics **7**, 355–369 (1936); (32) ibid. **7**, 179–188 (1936); (33) ibid. **8**, 376–386 (1938); (34) ibid. **10**, 422–429 (1940) [these Rev. **2**, 235]; (35) ibid. **9**, 174–180 (1939); (36) ibid. **9**, 238–249 (1939) [these Rev. **1**, 248]; (37) ibid. **10**, 14–17 (1940) [these Rev. **1**, 347]; (38) ibid. **11**, 182–187 (1941) [these Rev. **4**, 26]; (39) ibid. **11**, 341–353 (1942) [these Rev. **4**, 126]; (40) ibid. **12**, 283–290 (1945) [these Rev. **7**, 107]; (41) ibid. **11**, 395–401 (1942) [these Rev. **4**, 183]; (42) ibid. **11**, 306–307 (1942) [these Rev. **4**, 26]; (43) J. Animal Ecology **12**, 42–58 (1943). The volume also contains a portrait of the author, a biography by P. C. Mahalanobis [reprinted from Sankhyā **4**, 265–272 (1938)], and a subject index prepared by J. Tukey. The book is not consecutively paged, but contains about 650 pages.

Huron, Roger. Sur la répartition des décimales de rang donné dans les tables numériques. C. R. Acad. Sci. Paris **232**, 299–301 (1951).

The mean of n consecutive decimals in a fixed column of a table of a monotonic continuous $f(x)$ is evaluated approximately using the Euler-Maclaurin formula.

W. Feller (Princeton, N. J.).

Dynkin, E. B. On sufficient and necessary statistics for families of probability distributions. Doklady Akad. Nauk SSSR (N.S.) **75**, 161–164 (1950). (Russian)

The author studies the problem of estimating the parameter θ in families S of distributions on the line R, each distribution having a sectionally smooth density p. Call a function f inferior to a function g whenever $g(x)=g(y)$ implies $f(x)=f(y)$ (where the domains and ranges of f and g are arbitrary, except that f and g have the same domain), and call f and g equivalent whenever each of them is inferior to the other. A statistic (i.e., for the author, a function from R^n to R^m, where n and m are positive integers) is called trivial if it is equivalent in some nonempty open set to the identity map of R^n onto itself. A statistic is called necessary (for the fixed family S and the fixed integer n) if it is inferior to every sufficient statistic.

Let θ_0 be any fixed value of the parameter, and write, for each x in R and for each value of the parameter θ, $g_\theta(x)=\log p(x,\theta)-\log p(x,\theta_0)$. If the dimension of the linear space L generated by the functions g_θ together with the constant function 1 is $r+1$ ($1\leqq r\leqq\infty$), the number r is called the rank of the family S. The author's first theorem has two parts: (A) if $n<r$ and $n<\infty$, then every sufficient statistic (for S and n) is trivial; (B) if $\{1,\varphi_1,\cdots,\varphi_r\}$ is a basis of L, if $n\geqq r$, and if $\chi_i(x_1,\cdots,x_n)=\sum_{k=1}^{n}\varphi_i(x_k)$, $i=1,\cdots,r$, then the functions χ_i are functionally independent and the vector function $(\chi_1,\cdots,\chi_r)$ is a necessary and sufficient statistic (for S and n).

The author goes on to state a necessary and sufficient condition on the form of the densities p in order that S have finite rank, gives more detailed representation theorems for the densities in case the families obtained from S by translations and changes of scale (or both) do have finite rank, and concludes by mentioning several examples.

P. R. Halmos (Chicago, Ill.).

Rao, C. Radhakrishna. A note on unbiased and minimum variance estimates. Calcutta Statist. Assoc. Bull. **3**, no. 9, 36 (1950).

Some elementary properties of unbiased estimates. For example: If T is an unbiased estimate with minimum variance and T_1, T_2 are two unbiased estimates (of the same quantity) with common variance, then the correlation between T_1 and T_2 is $\geqq 2e-1$, where $e=(\text{var } T)/(\text{var } T_1)$.

W. Hoeffding (Chapel Hill, N. C.).

Ghosh, M. N. Rank-weighted mean and its use in descriptive statistics. Calcutta Statist. Assoc. Bull. **3**, no. 9, 32–35 (1950).

A measure of central tendency W and a consistent estimate w of W are defined and called rank-weighted means. For symmetric populations W agrees with the mean. For a sample $(x_1,x_2,\cdots,x_n)$, $w=\sum a_i x_i/n(n-1)(n-2)$, where $a_i=(r-1)(n-r)$ and r is the rank of x_i when the observations are arranged in increasing order. For samples from normal populations w is about as efficient as the median. The bias due to contamination by an observation from a different population is stated to be less than for the arithmetic mean under certain conditions. The proof is incomplete and the statement is wrong for certain bimodal distributions.

H. Chernoff (Urbana, Ill.).

Arley, N., and Hald, A. On the mean successive difference estimate of dispersion. Mat. Tidsskr. B. 1950, 86–89 (1950).

Let x_j be independent normal variables with mean μ and variance σ^2, and put $d_k=|x_k-x_{k+1}|$. The statistic $s'=\pi^{\frac{1}{2}}(d_1+\cdots+d_n)/2n$ is an unbiased estimate of σ. The authors find that Var $(s')\approx\sigma^2(0.82645n^{-1}-0.25565n^{-2})$.

W. Feller (Princeton, N. J.).

Ehrenberg, A. S. C. Estimation of heterogeneous error variances. Nature 166, 608 (1950).

Let $X_{11}, \cdots, X_{pq}$ be mutually independent normal chance quantities where X_{ij}'s mean and variance are $\alpha_i+\beta_j$ and σ_j^2, respectively, and the values of α_i, β_j, and σ_j^2 are unknown ($i=1, \cdots, p$; $j=1, \cdots, q$). The variance σ_j^2 is to be estimated on the basis of pq observed values of $X_{11}, \cdots, X_{pq}$, respectively. If p or q is small, the maximum-likelihood estimator is not satisfactory. Two unbiased estimators of σ_j^2 are presented. *D. F. Votaw, Jr.* (New Haven, Conn.).

Greenwood, Robert E., and Glasgow, Mark O. Distribution of maximum and minimum frequencies in a sample drawn from a multinomial distribution. Ann. Math. Statistics 21, 416–424 (1950).

Let $\{n_1, \cdots, n_k\}$ have a multinomial distribution with the probability that $\{n_1=r_1, \cdots, n_k=r_k\}$ given by $(N!/r_1!\cdots r_k!)p_1^{r_1}\cdots p_k^{r_k}$, where $\sum_{i=1}^k p_i=1$ and $\sum_{i=1}^k r_i=N$. The authors evaluate the mean and the variance of the statistics max $(n_1, \cdots, n_l)$ and min $(n_1, \cdots, n_l)$ for the binomial case where $l=k=2$, and for the special trinomial case where $p_1=p_2$ and $l=2$, $k=3$. Exact results were obtained for the binomial case and approximate results based on the use of the normal approximation to the multinomial distribution were found for both the binomial and the special trinomial cases. *E. Paulson* (Seattle, Wash.).

Zeigler, R. K. A note on the asymptotic simultaneous distribution of the sample median and the mean deviation from the sample median. Ann. Math. Statistics 21, 452–455 (1950).

Let $x_1, \cdots, x_{2k+1}$ be a sample of $2k+1$ independent observations on a random variable with probability density $f(x)$ which has a finite second moment, and which is continuous at $x=\theta$ with $f(\theta)\neq 0$, where θ is the population median. The author shows that the joint distribution of the sample median and the mean deviation from the sample median is asymptotically normal, and gives the asymptotic means, variances, and correlation. *E. Paulson.*

Rao, C. Radhakrishna. A note on the distribution of $D^2_{p+q}-D^2_p$ and some computational aspects of D^2 statistic and discriminant function. Sankhyā 10, 257–268 (1950).

The author considers the distribution of $(c/f)(D^2_{p+q}-D_p^2)$, where D^2_{p+q} and D_p^2 are the squares of the sample distances between two populations based on $p+q$ and p characters, respectively, f the number of degrees of freedom in estimating the variances and covariances entering into the distances, and $c=n_1n_2/(n_1+n_2)$, where n_1 and n_2 are the sample sizes in the two populations, under the null hypothesis that the true distances based on p and on $p+q$ characters are the same. Several approximations are found to the distribution given in an earlier paper of the author [Sankhyā 9, 343–366 (1949); these Rev. 11, 448]. The author concludes with a note on transforming a set of correlated variables into an uncorrelated set by a recursive method. *K. J. Arrow* (Stanford University, Calif.).

Gumbel, E. J., and Keeney, R. D. The extremal quotient. Ann. Math. Statistics 21, 523–538 (1950).

The extremal quotient q of a sample is the absolute value of the ratio of the largest to the smallest observation. The authors study the asymptotic distribution of q when the underlying distribution $F(x)$ is continuous, unlimited, and symmetric with respect to the origin. Particularly, if $F(x)$ is of the exponential type, and if the parameters u and α are defined by $F(u)=1-1/n$, $\alpha=nF'(u)$, then the variable $t=\alpha u(q-1)$ has the asymptotic distribution function $(1+e^{-t})^{-1}$. If $F(x)$ is of the Cauchy type, i.e., if $1-F(x)\sim Ax^{-k}$ $(x\to\infty)$, then $t=k\log q$ has the asymptotic distribution mentioned.

G. Elfving (Ithaca, N. Y.).

Chand, Uttam. Distributions related to comparison of two means and two regression coefficients. Ann. Math. Statistics 21, 507–522 (1950).

The work of Welch [Biometrika 29, 350–362 (1938)] and Hsu [Statist. Res. Mem. London 2, 1–24 (1938)] on the Behrens-Fisher problem is here extended to the one-sided case of this problem and to the hypothesis of equality of two regression coefficients. The author also investigates how best to utilize approximate information concerning the ratio of the two unknown variances and recommends Welch's statistic for this purpose. *E. L. Lehmann.*

Bahadur, Raghu Raj, and Robbins, Herbert. The problem of the greater mean. Ann. Math. Statistics 21, 469–487 (1950); 22, 310 (1951).

Knowing that two samples (not necessarily of the same size) are from normal populations with means m_1, m_2 and common variance σ^2, how should one decide, on the basis of the samples, which mean, if either, is the greater? According to the minimax theory the answer depends on what set of parameter triples $(m_1, m_2; \sigma^2)$ is regarded as possible and on what economic loss will be incurred by incorrect decisions. Minimax problems arising thus are discussed and many are solved. It is pointed out that some of the results attained are at variance with naive misapplication of the conventional theory of testing hypotheses.

L. J. Savage (Chicago, Ill.).

Walsh, John E. Some estimates and tests based on the r smallest values in a sample. Ann. Math. Statistics 21, 386–397 (1950).

The author investigates the efficiency of nonparametric order statistics for estimating the $100p$ % point of a probability distribution by comparing it with estimates based on the assumption of a normal distribution. The comparison is made for (a) all the n observations are available with σ unknown, (b) only the $r=pn+O(n^{\frac{1}{2}})$ smallest observations are available with σ known, and (c) only the $r=\beta n+O(n^{\frac{1}{2}})$ observations are available $(\beta>p)$ with σ unknown. For cases (b) and (c) the nonparametric estimate appears to be fairly efficient for small values of p. *E. Paulson.*

Dixon, W. J. Analysis of extreme values. Ann. Math. Statistics 21, 488–506 (1950).

An analysis is made of the efficiencies of various tests for detecting contaminated observations, i.e., observations from a population other than the one theoretically being sampled. Two models are discussed. In each model the population being sampled is normal with mean μ and variance σ^2, $N(\mu, \sigma^2)$. In one model the contaminating observations come

from the population $N(\mu+\lambda\sigma, \sigma^2)$ (location error) and in the other model from $N(\mu, \lambda^2\sigma^2)$ (scale error).

H. Chernoff (Urbana, Ill.).

Hemelrijk, J. A family of parameterfree tests for symmetry with respect to a given point. II. Nederl. Akad. Wetensch., Proc. **53**, 1186–1198 = Indagationes Math. **12**, 419–431 (1950).

[For part I see the same vol., 945–955 = Indagationes Math. **12**, 340–350 (1950); these Rev. **12**, 37.] A two-stage test is proposed. Let $x_1, \cdots, x_n$ be positive deviations from the given point and $y_1, \cdots, y_m$ be absolute values of the negative deviations. Using the ordinary sign test one first tests the null hypothesis that m does not differ significantly from n. If this is not rejected, then one tests whether the x's and y's may be regarded as coming from the same population. For this latter test one may use any of the two-sample tests depending on what alternatives to symmetry he is particularly interested in detecting.

A. M. Mood (Santa Monica, Calif.).

Janko, Jaroslav. Advances in the theory of non-parametric tests in statistical inference. Časopis Pěst. Mat. Fys. **74** (1949), 62–74 (1950). (Czech. English summary)

Summary of an expository lecture.

Armitage, P. Sequential analysis with more than two alternative hypotheses, and its relation to discriminant function analysis. J. Roy. Statist. Soc. Ser. B. **12**, 137–144 (1950).

Wald's sequential probability ratio test [Sequential Analysis, Wiley, New York, 1947; these Rev. **8**, 593] is extended in a straightforward manner to problems where the number of simple hypotheses is more than two (but still finite). The same inequalities that Wald obtained to relate the various errors of the test to the constants determining it, generalize to this case. However, as the author shows by the example of the binomial where there are three hypotheses (i.e., $H_i: p = p_i, i = 1, 2, 3$), some of these inequalities may be far from sharp. Other examples and the application of the procedure to discriminatory analysis are discussed very briefly. *D. G. Chapman* (Seattle, Wash.).

Hodges, J. L., Jr. The choice of inspection stringency in acceptance sampling by attributes. Univ. California Publ. Statist. **1**, 1–14 (1949).

The author discusses the theoretical implications of setting up an attribute inspection plan as follows: Choose an indifference quality and let probability of acceptance be 0.5 at that quality; then maximize the slope (at least locally) of the characteristic curve through the chosen point. Two examples are described, one in which the population is normal, and the other in which it is Poisson; inspection is on a "go-no go" basis. *A. M. Mood.*

Pearson, E. S. On questions raised by the combination of tests based on discontinuous distributions. Biometrika **37**, 383–398 (1950).

It is in practice often necessary to combine the results of different experiments in order to test a given hypothesis. R. A. Fisher [Statistical Methods for Research Workers, 4th ed., Oliver and Boyd, Edinburgh, 1932, § 21.1] was the first to propose the use of the probability integral transformation for this purpose. If $p(x)$ is the density function of the random variable X and $Y = \int_{-\infty}^{X} p(x)\,dx$, then $-2 \log Y$ has the χ^2 distribution with 2 degrees of freedom. Thus the results of k independent experiments can be combined and the hypothesis can be tested using χ^2 with $2k$ degrees of freedom as a test statistic. The author also considers two other similar tests. All these methods, however, give exact results only if the distributions of the X's are all continuous. The author discusses in detail a method of making discrete random variables continuous by the addition of a continuous random variable, so that Fisher's method becomes directly applicable. The method is illustrated on data that arose from statistical practice. The main purpose of the paper is an impartial discussion of the arguments that have been made for and against the introduction of such postexperimental random elements. To the reviewer the arguments against the procedure seem to be largely based on superstitions. He cannot see what difference it should make whether random factors are introduced in the course of the arrangement of an experiment or whether they are introduced after its completion as a computational device. The only criterion in both cases is the power function. Unfortunately, however, from the discussion of the author it appears that the effect on the power function due to this postexperimental random element is not sufficiently known. His discussion therefore remains necessarily inconclusive.

H. B. Mann (Columbus, Ohio).

Lancaster, H. O. The exact partition of χ^2 and its application to the problem of the pooling of small expectations. Biometrika **37**, 267–270 (1950).

A technique of separating the chi-square of a contingency table into a sum of chi-squares with one degree of freedom is applied to the case when the expected frequencies in certain cells are small. Then these cells may frequently be conveniently pooled so that the chi-square is the sum of chi-squares with one degree of freedom plus a chi-square involving the pooled cells. *H. Chernoff* (Urbana, Ill.).

Paull, A. E. On a preliminary test for pooling mean squares in the analysis of variance. Ann. Math. Statistics **21**, 539–556 (1950).

In any complex analysis-of-variance experiment several models are possible depending on whether interaction terms are assumed to exist or not, etc. A commonly suggested procedure is to test for the "significance" of such interaction terms, as a preliminary step; if these terms are "insignificant," the model is changed in the belief that a more powerful test of the primary hypothesis is obtained. The author considers the changes, caused by such a preliminary test, in the Type I error and the power of the final test. He obtains inequalities and bounds for these quantities, by a geometrical and analytical approach, and he also gives extensive numerical illustrations of the many possible situations. He includes a number of recommendations, based on the results of his numerical computations, that are of interest to the practical statistician. *D. G. Chapman.*

Finney, D. J. The fractional replication of factorial experiments—a correction. Ann. Eugenics **15**, 276 (1950).

The author calls attention to an incorrect table in an earlier publication [same Ann. **12**, 291–301 (1945); these Rev. **7**, 213] and gives the correct entries.

Vaswani, Sundri. Assumptions underlying the use of the tetrachoric correlation coefficient. Sankhyā **10**, 269–276 (1950).

An example of a nonnormal bivariate distribution of X and Y with normal marginal distributions of X and Y,

respectively, is given. It is argued that no monotonic transformations $x=f_1(u)$ and $y=f_2(v)$ will give random variables U and V with a bivariate normal distribution. A $2\times2\times2$ table of sample frequencies is considered to arise from dichotomization on each of three random variables with a trivariate normal distribution with zero means and unit variances. Two methods of estimating the dichotomization points and correlations are given, and tests of the hypothesis that the table arose in this manner are given.

T. W. Anderson (New York, N. Y.).

Amato, Vittorio. Sulle correlazioni parziali. Statistica, Milano **10**, 341–350 (1950).

This is largely an expository article giving determinantal formulae for partial coefficients of correlation. In addition, certain relations due to Kendall [J. Roy. Statist. Soc. (N.S.) **104**, 281–283 (1941); these Rev. **3**, 173] are slightly generalized. *E. Lukacs* (Washington, D. C.).

Kermack, K. A., and Haldane, J. B. S. Organic correlation and allometry. Biometrika **37**, 30–41 (1950).

The authors discuss first the problem of a suitable measure of linear correlation in the case of organic data. The traditional regression lines are not ideal since there may be no determinate choice of independent as against dependent variable. The major axis of the correlation surface, first suggested by Adcock [Analyst **5**, 53–54 (1878)] is invariant under rotation and reflection (often a negligible advantage), but not under change of scale. The suggestion of H. E. Jones [Metron **13**, no. 1, 21–30 (1937)] and G. Teissier [Biometrics **4**, 14–53 (1948)] of considering the major axis of variates expressed in standard measure (the "reduced major axis") is appropriate and simple to use. The corresponding variance is here computed in the general, and in certain special cases. The authors then consider three analogous measures when the trend is more appropriately represented by a power law (allometric growth of Huxley [Problems of Relative Growth, MacVeagh, London, 1932]). Various formulas are secured. The accuracy of most of the formulas is found to be not very sensitive to a moderate deviation from normality of the distribution of the logarithms of the variates. Appendices examine in detail (6 tables) an illustrative example (338 specimens of Micraster cor-anguinan from the chalk of Northfleet, Kent) and give a summary of the main formulas developed in this paper. *A. A. Bennett.*

Johnson, N. L. Time series. J. Inst. Actuaries Students' Soc. **10**, 13–23 (1950).

From a lecture given in March, 1950.

Gibson, Gordon D. A rapid method for ascertaining serial lag correlation. Biometrika **37**, 288–307 (1950).

The author is interested in matching series of tree ring width measurements, and to this end constructs a routine method for the rapid measurement of the lag correlations between two series. Instead of the usual second order moment correlation he uses a first order moment correlation, based upon absolute values of sums and differences. In practice the two series are recorded as bar graphs on film and by the use of these transparencies and a photoelectric cell the first order moment lag correlations are relatively quickly obtained. A large portion of the paper is devoted to the correction of errors due to the local variation of mean and variance in the series. *P. Whittle.*

Solodovnikov, V. V. Analysis and synthesis of tracking and regulating systems subject to stationary random disturbances. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 1648–1679 (1950). (Russian)

This paper is an expository review of the theory of linear prediction and filtering. *E. N. Gilbert.*

Moyal, J. E. The momentum and sign of fast cosmic ray particles. Philos. Mag. (7) **41**, 1058–1077 (1950).

The author derives the probability distribution of the angular deviations along the track of magnetically deflected and multiply scattered particles. The physical conditions and approximations introduced cannot be described in a few lines. Maximum likelihood estimates are given and their efficiency is discussed. *W. Feller* (Princeton, N. J.).

Mathematical Biology

***Mather, K. Biometrical Genetics. The Study of Continuous Variation.** Dover Publications, Inc., New York, N. Y., 1949. ix+158 pp. $3.50.

An account of the problems arising in the study of quantitative characters determined by numerous genes, and of the methods, experimental and statistical, which have been devised for their solution. Among the more important topics discussed are: the problem of scaling and criteria for scaling; the determination of the components of variation, under various experimental conditions; the problems of linkage; the estimation of the numbers of factors involved.

C. P. Winsor (Baltimore, Md.).

Geppert, Maria Pia. Biologische Gesetze im Lichte der Mathematik. Mitteilungsblatt Math. Statist. **1**, 145–166 (1 plate) (1949).

An exposition of work by von Pfaundler [Münchener Mediz. Wochenschr. **89**, 115–122 (1942)] on sex ratio in disease, by de Rudder [Naturwissenschaften **31**, 577–584 (1943)], and von Schelling [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1944**, no. 5; these Rev. **8**, 479] on the Weber-Fechner law, and by von Schelling [ibid., no. 6; these Rev. **8**, 214] on all-or-none phenomena. *C. P. Winsor.*

Bradistilov, G. The rôle of mathematical analysis in problems of biology. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **37**, 117–130 (1941). (Bulgarian)

Luneburg, Rudolf K. The metric of binocular visual space. J. Opt. Soc. Amer. **40**, 627–642 (1950).

This last paper of the author, who died in 1949, was not written for publication, but has been edited by P. Boeder and others. As in preceding work [Mathematical Analysis of Binocular Vision, Princeton University Press, 1947; Studies and Essays Presented to R. Courant on his 60th Birthday, pp. 215–240, Interscience, New York, 1948; these Rev. **9**, 49, 735, 296], the aim is to reduce the study of binocular perception to the geometry of a Riemannian space. In the present paper the view is taken that binocular perception of a single point does not differ from monocular observation; there is then no absolute localisation, and the perception of distances between points becomes the important thing, together with the sensing of apparently straight lines. The author introduces a psychometric dis-

tance function $D(P_1, P_2)$ for binocular visual space, expressible in terms of physical coordinates and subject to conditions including the following: the triangle inequality, $(P_1, P_2) > (P_3, P_4)$ (where these are sensed distances) implies $D(P_1, P_2) > D(P_3, P_4)$; apparent collinearity implies $D(P_1, P_2) + D(P_2, P_3) = D(P_1, P_3)$, and conditions of compactness and convexity. When these are combined with the assumption that localisation is not absolute (a condition of homogeneity) the conclusion is reached that $D(P_1, P_2)$ is given by the formula

$$2(-K)^{-\frac{1}{2}} \sin [\tfrac{1}{2}(-K)^{\frac{1}{2}} D/C] = [(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2 + (\zeta_1-\zeta_2)^2]^{\frac{1}{2}} (1+K\rho_1^2/4)^{-\frac{1}{2}} (1+K\rho_2^2/4)^{-\frac{1}{2}},$$

where $\rho_n^2 = \xi_n^2+\eta_n^2+\zeta_n^2$, K and C are real constants, and ξ, η, ζ Cartesian coordinates in a Euclidean space. Helmholtz' horopter curves, the distance alleys of Blumenfeld, and the parallel alleys of Hillebrand are discussed; the Blumenfeld experiments give direct demonstration that the space of binocular vision is a Lobachevski space. Further comparisons of theory with experiment are discussed and the paper ends with a brief section on the problem of visual perceptions of moving objects or of a moving observer; it is suggested that a space-time representation would be closely related to the space-time of special relativity, the extreme velocity of eye motions playing a role analogous to the velocity of light. *J. L. Synge* (College Park, Md.).

Haldane, J. B. S. Some statistical problems arising in genetics. J. Roy. Statist. Soc. Ser. B. 11, 1–9; discussion, 9–14 (1949).

Illustrations of problems of sampling, of significance testing, and of distribution functions. *C. P. Winsor.*

Bailey, Norman T. J. A simple stochastic epidemic. Biometrika 37, 193–202 (1950).

The author sets up a stochastic model of an epidemic, and evaluates various mean values. For example he evaluates the expectation, variance, and coefficients of skewness and kurtosis of the time until the whole population of susceptibles is infected. *J. L. Doob* (Urbana, Ill.).

Rapoport, Anatol. Contribution to the probabilistic theory of neural nets. II. Facilitation and threshold phenomena. Bull. Math. Biophys. 12, 187–197 (1950).

Continuing a previous paper [same vol., 109–121 (1950); these Rev. 12, 39] the author assumes that a neuron will fire when impinging impulses occur with a threshold frequency, the response frequency being then $f(x) = x/(h+[\delta x])$, where x is the stimulus frequency, h the threshold frequency, δ the refractory period, and the brackets denote the greatest integer. He then considers the conditions under which $f(x)$ resembles an "all-or-none" output (a single-step function) in case impulses impinge with complete regularity, and again in case they have a Poisson distribution.

A. S. Householder (Oak Ridge, Tenn.).

Pinney, Edmund. On a mathematical theory of the reaction of cells to X-ray irradiation. Bull. Math. Biophys. 12, 199–206 (1950).

Solution for $t \geqq 0$ of the equation

$$dx(t)/dt = b[x_0 - x(t-u)] - ax(t),$$

where a is a positive constant for $0 < t < T$ and is zero for $t > T$, b is a positive constant for $t > u$ and is zero for $0 < t < u$. Method: Laplace transformation and Heaviside's generalized expansion theorem. A discussion of the oscillatory character of the solutions is included. *I. Opatowski.*

van Iterson, G., Jr. The utility of probability paper in studying the statistics of biological variations. Statistica, Rijswijk 4, 129–157 (1950). (Dutch. English summary)

Mathematical Economics

***Koopmans, Tjalling C., editor. Statistical Inference in Dynamic Economic Models.** Cowles Commission Monograph No. 10. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. xv+438 pp. $6.00.

This is the long awaited volume in which the Cowles Commission group report the main results of their research into the statistical analysis of simultaneous economic relationships. The volume contains 19 papers, written by 7 members of the group and 4 associated authors. Because of the close connexion between the papers, the entire volume will be covered by a single review.

The individual papers, as grouped in the book, are as follows. (I) J. Marschak, Statistical inference in economics: an introduction (pp. 1–50). (II) T. C. Koopmans, H. Rubin, and R. B. Leipnik, Measuring the equation systems of dynamic economics (pp. 53–237). Problems of identification: (III) A. Wald, Note on the identification of economic relations (pp. 238–244); (IV) L. Hurwicz, Generalization of the concept of identification (pp. 245–257); (V) T. Haavelmo, Remarks on Frisch's confluence analysis and its use in econometrics (pp. 258–265). Problems of structural and predictive estimation: (VI) L. Hurwicz, Prediction and least squares (pp. 266–300); (VII) T. C. Koopmans, The equivalence of maximum-liklihood and least-squares estimates of regression coefficients (pp. 301–304); (VIII) A. Wald, Remarks on the estimation of unknown parameters in incomplete systems of equations (pp. 305–310); (IX) T. W. Anderson, Jr., Estimation of the parameters of a single equation by the limited-information maximum-liklihood method (pp. 311–322). Problems of computation: (X) H. Hotelling, Some computational devices (pp. 323–325). Time series problems. (a) Trend and seasonability: (XI) L. Hurwicz, Variable parameters in stochastic processes: trend and seasonality (pp. 329–344); (XII) H. B. Mann, Nonparametric tests against trend (pp. 345–351); (XIII) R. L. Anderson, Tests of significance in time-series analysis (pp. 352–355). (b) Estimation problems: (XIV) H. Rubin, Consistency of maximum-liklihood estimates in the explosive case (pp. 356–364); (XV) L. Hurwicz, Least-squares bias in time series (pp. 365–383). (c) Continuous stochastic processes: (XVI) T. C. Koopmans, Models involving a continuous time variable (pp. 384–389). Specification of hypotheses: (XVII) T. C. Koopmans, When is an equation system complete for statistical purposes? (pp. 393–409); (XVIII) L. Hurwicz, Systems with nonadditive disturances (pp. 410–418); (XIX) H. Rubin, Note on random coefficients (pp. 419–521). The volume ends with a list of references and an index. Papers will be referred to by number in the review.

Preceded by abstracts and partial releases, the report has been announced since 1945, and it is stated in the expository Introduction (I, p. 5) that the material was prepared in 1944–1946, and that the manuscript was completed early in 1947.

To illustrate the approach by a simple example (quoted from a recent lecture by Koopmans), let x, x_1, x_2, y_1, y_2, y_3 be variables in the sense of probability theory. Let

$$(R) \qquad x=\beta_1 y_1+\beta_2 y_2+\beta_3 y_3+u$$

be the ordinary least square regression of x on y_1, y_2, y_3, with u for regression residual. On the other hand, let

$$(S) \qquad \begin{array}{l|l} \alpha_{11}x_1+\alpha_{12}x_2 & +\beta_{11}y_1+\beta_{12}y_2+\beta_{13}y_3=u_1 \\ \alpha_{21}x_1+\alpha_{22}x_2 & +\beta_{21}y_1+\beta_{22}y_2+\beta_{23}y_3=u_2 \end{array}$$

be two a priori relationships between the variables, where u_1, u_2 (the disturbances) indicate the relations to be approximate. By the properties of regression residuals, u in (R) is uncorrelated with y_1, y_2, y_3. In (S) the x_i are separated from the y_i by a vertical, to indicate the a priori assumption that the u_i are uncorrelated with the y_i, but in general not with the x_i. The authors describe the x_i in (S) as jointly dependent on the predetermined variables y_i. The u_i are allowed to be mutually intercorrelated, say with covariances σ_{ik}. Clearly the ordinary regression method does not in general apply to (S), for in the resulting regression of x_1 on x_2, y_1, y_2, y_3 the residual u would be uncorrelated with x_2, and similarly in the regression of x_2 on x_1, y_1, y_2, y_3.

The approach (S) extends to any number of variables $x_1, \cdots, x_g$ and $y_1, \cdots, y_h$. If there are g relations in (S), as many as jointly dependent variables, (S) is called complete. The main theme of the volume is the estimation of the parameters α_{ik}, β_{ik}, σ_{ik} in complete systems of jointly dependent variables. The maximum-likelihood principle is taken as basis for the estimation.

Closely tied up with the approach (S) is the notion of identifiable relations. Slightly rephrasing the definition given by the authors, let P be the joint probability distribution that the system considered specifies for the observable variables. Then a relation is identifiable if its parameters can be expressed uniquely in terms of P. Now, as seen immediately, a relation will not be identifiable unless (S) is subjected to some kind of a priori restrictions. Main attention is paid to restrictions of the type

$$(*) \qquad \alpha_{ik}=0 \quad \text{or} \quad \beta_{ik}=0;$$

a more general case considered allows linear relations between the parameters. The authors attribute [II, p. 69] the first systematic discussion of identifiability to R. Frisch [unpublished memo, 1938].

It is claimed explicitly [XVI, p. 384] that the simultaneous-equations approach (S) constitutes an important advance over the single-equation approach (R) in the measurement of economic relations. At the same time the authors stress that the new methods are subject to certain shortcomings in the present stage. The claim is a bold one, however, in view of its wide scope, for even if the approach (R) by long experience works fairly well in several fields of economic analysis, better methods would be highly welcome, not to speak of their potential applications in other social sciences. Now in the opinion of the reviewer, it may be stated at once that further material, theoretical as well as empirical, must be available before a definite appraisal of the approach (S) can be attempted. This volume is purely theoretical, and numerical calculations are reported only for illustration of computation devices [for applications, see L. R. Klein, "Economic Fluctuations in the United States, 1921–1941," Cowles Commission Monograph No. 11, Wiley, New York, 1950]. Yet as regards the theoretical side there is, as pointed out below, some obscurity in the causal interpretation of the approach (S). For another thing, the new methods in their present form are based on the hypothesis that disturbances u_i successive in time are independent. It is admitted [XVI, p. 384] that in general this hypothesis is not realistic in economic data; but to allow for interdependence seems to be extremely difficult [XVI, p. 387]. The decisive test of the new methods, however, will lie in the empirical applications, and in particular in the success of resulting predictions, yet it will take long before anything definite can be said in this respect. Anyhow, advantages of (R) to be outweighed in the final comparison are flexibility in its adaptation to specific situations, and simplicity in the numerical calculations.

The dynamic economic systems considered are of type (S), with the endogenous variables x_i referring to the same time-point t, the y_i being either exogenous variables or lagged endogenous variables [II, p. 59]. The endogenous variables are to be explained by the system, whereas the exogenous variables are outside factors influencing the system, e.g., meteorological factors. The observations of the x_i, y_i are assumed to be equidistant in time, say x_{it}, y_{it} $(t=1, \cdots, T)$. Assuming the matrix $\{\alpha_{ik}\}$ to be nonsingular, let (S) be solved for the x_i, say

$$(U) \qquad x_i=\kappa_{i1}y_1+\cdots+\kappa_{ih}y_h+u_i^*, \qquad i=1, \cdots, g,$$

where the u_i^* are linear in the u_i. The system (U) is called the reduced form of (S). An important point stressed by Hurwicz [VI, p. 269; cf. also II, p. 117] is that if we wish to predict one or more x_i this can be done from (U), estimating $\kappa_{i1}, \cdots, \kappa_{ih}$ by a least square regression of type (R).

This simple treatment of the prediction problem is subject to the qualification that no future change is expected in the system (S), or, in the authors' terminology, that the structural relations remain the same. If on the other hand the structural relations have been identified and estimated, expected changes in the structure can be taken into account in the prediction. As to the subject-matter interpretation of structural relations, it appears from remarks by Koopmans [XVII, pp. 394, 405; cf. also II, p. 60] that they are intended as causal relationships, and so their estimation is of fundamental importance in itself, even disregarding prediction problems. Now unfortunately there is an obscure point in this issue. To explain, let us consider the regression (R) as applied to the observations of an ordinary experiment, taking for x the effect variable and for the y_i the cause variables under controlled variation. Such a regression may be regarded as the prototype of statistical relations that allow a causal interpretation; we note (i) the causal dependence is here unilateral in the sense that it is directed from the cause variables to the effect variable, not the other way around, and (ii) from the very design of the experiment considered it is obvious which is the effect variable and which are the cause variables. As to the application of (R) to economic and other nonexperimental data we note: (iii) The unilateral dependence is here not a priori obvious; its justification or possible rejection is a question of subject-matter considerations from case to case. Now the system (S) contains several dependent variables. This in itself is nothing peculiar from the experimental viewpoint, for the jointly dependent variables x_i may be g measurable aspects of the effect phenomenon. A natural extension of the approach (R) would then be to estimate the joint probability distribution of the x_i as conditioned by the y_i. Assuming normal distribution, this would only require the estimation of the conditioned moment matrix of the x_i.

[This problem appears equivalent to the estimation of the reduced form (*U*).] But the approach (*S*) goes further: The analysis is extended to the interdependence between the effect variables x_i, these being assumed to satisfy linear relations (*S*) that involve these same x_i. This situation is not the same as that mentioned in (i) and (iii) above, for if we single out one effect variable in each relation of (*S*) there will in general be one relation that has x_i as effect and x_h as cause variable, and another that has x_h as effect and x_i as cause. It is not clear to the reviewer how the relations (*S*) are to be interpreted under such circumstances. On the one hand there is a reference [XVI, p. 384] to a paper by Koopmans [J. Amer. Statist. Assoc. **40**, 448–466 (1945); these Rev. **7**, 215] where unilateral dependence in the sense of (i) is rejected. If so, which subject-matter interpretation is then intended for the apparent "bilateral" dependence in (*S*)? On the other hand it is stressed that (*S*) should be regarded as approximate [XVI, p. 386], the ideal being a system that involves time lags with t as a continuous variable, and the models used to illustrate this point [XVI, pp. 388 ff.] can in point of principle be interpreted in terms of unilateral dependence. In any case, very little is known about the bias that creeps in when treating t as a discrete variable [cf. also Bentzel and the reviewer, Skand. Aktuarietidskr. **29**, 95–114 (1946); these Rev. **8**, 216].

[Reviewer's note. Suppose the x_i in (*S*) can be ordered so that for $i=1, \cdots, g$ every x_i is unilaterally dependent on $x_{i-1}, \cdots, x_1$ and on the y_i; in symbols

$$(V) \quad \alpha_{i1}x_1+\cdots+\alpha_{ii}x_i+\beta_{i1}y_1+\cdots+\beta_{ih}y_h=u_i, \quad i=1, \cdots, g,$$

with $\alpha_{ii}\neq 0$. We see (i) that (*V*) is the most general subclass of (*S*) that evades bilateral dependence, and (ii) that the dynamic approach known as economic sequence analysis (for the illustration of which Tinbergen devised his "arrow graph" [cf. Bentzel and the reviewer, loc. cit.]) is based on recursive systems that in the linear case are precisely of type (*V*). However, to judge from a footnote remark [I, p. 35] the theory reviewed aims at systems (*S*) that do not reduce to the recursive form (*V*).]

The main paper [no. II] has 4 sections. Section 1 is a survey of problems and results. Section 2 treats the problems of identification, under alternative hypotheses about (i) the type of the a priori restrictions on (*S*), and (ii) the identifiability, nonidentifiable systems being studied with respect to identifiable linear functions of the parameters. The analysis thus treating several alternative cases, the review, for considerations of space, can mention one main result only [theorem 2.2.1, p. 78]: Suppose that the a priori restrictions are of type (*), and that the u_i are independent of the y_i. Let A be the $g\times(g+h)$ matrix of the α_{ik}, β_{ik} in (*S*), and A_i the submatrix formed by those columns of A for which (*) indicates the element in the ith row to equal 0. Then the ith relation in (*S*) is identifiable if and only if A_i is of rank $g-1$. [Hence, in (*U*) all relations are identifiable, whereas in (*V*) all relations except the 1st are nonidentifiable.] Sections 3–4 develop, under the alternative assumptions, the maximum-likelihood (m.l.) equations for the parameter estimates. The m.l. equations being essentially nonlinear, the authors state [II, p. 122] that they have not succeeded in finding general conditions for the existence of at least one real solution. Extending the Mann-Wald theorems on simultaneous equations [Econometrica **11**, 173–220 (1943); these Rev. **5**, 129; cf. also Rubin, Ann. Math. Statistics **19**, 598 (1948); **20**, 137 (1949)], the authors discuss the large-sample properties of the m.l. estimates, and give asymptotic formulas for their sampling variances and covariances. For the numerical computation of the m.l. estimates, section 4 describes iterative procedures, which the authors refer to as related to Hotelling's method for inverting a matrix [cf. X below].

Of the remaining papers some are expository, commenting on or giving abstracts of contributions published before. Such are Haavelmo's paper, no. V [cf. Econometrica **11**, 1–12 (1943); **12**, Supplement (1944); these Rev. **4**, 220; **6**, 93], no. IX [cf. Anderson and Rubin, Ann. Math. Statistics **20**, 46–63 (1949); these Rev. **10**, 464], no. X [cf. Hotelling, ibid. **14**, 1–34, 440–441 (1943); these Rev. **4**, 202; **5**, 245], and no. XII [cf. Mann, Econometrica **13**, 245–259 (1945); these Rev. **7**, 21].

Other papers bring novel general results on related problems. Thus, in no. III, Wald specifies the notion of identifiable parameters by distinguishing between the cases when they are uniquely defined, and when their admissible values are finite or infinite in number. In no. IV, Hurwicz gives a lucid interpretation of identifiability by extending the definition to systems that are not necessarily linear or even parametric; in a concluding remark he states [p. 257] that in practice we are not often certain about the proper choice of identifying assumptions such as (*). In no. VI, Hurwicz first considers prediction in terms of the reduced form (*U*), as already mentioned. He then proceeds to examine the formal relationships between three general types of estimate, viz., maximum-likelihood (m.l.), least square (l.sq.), and unbiased linear estimates with minimum sampling variance, so-called Markov estimates. Distinguishing between absolutely and conditionally unbiased Markov estimates, Hurwicz shows that the former, if existing, are identical with the l.sq. estimates. Further, he brings out the relations between l.sq. and m.l. estimates by an analysis similar to the theorem of Bentzel and the reviewer [loc. cit., p. 106] that under general assumptions the two methods in question give identically the same estimates when applied to systems of type (*V*). Hurwicz's arguments on this latter topic are simplified by Koopmans in no. VII. In no. XVII, Koopmans has a related theorem that gives conditions for (*S*) to contain a subsystem that is complete from the viewpoint of m.l. estimation.

Third and last, some papers discuss related approaches and possible extensions, with emphasis not so much on giving definite general results as on pointing out problems for further research. Thus in no. VIII, Wald discusses the estimation problems for systems (*S*) that are incomplete. For the analysis of trend and seasonal variation, Hurwicz in no. XI considers some tentative m.l. approaches based on linear hypotheses. In no. XIII, R. L. Anderson briefly comments on the use of variance analysis for separating trend and seasonal components [for a related study, see Hald, Thesis, University of Copenhagen, 1948; these Rev. **9**, 363]. In no. XIV, Rubin shows that in the nonstationary Markov process $x_t=\rho x_{t-1}+u_t$, $|\rho|\geqq 1$, the m.l. estimate for ρ is asymptotically consistent. In no. XV, Hurwicz points out that there are cases where the l.sq. and the m.l. estimates are biased for any finite-sized sample, and evaluates the bias for certain special cases. Nos. XVI–XVII by Koopmans have already been mentioned; no. XVII merits special attention for its clarifying comments on application questions. The two final papers deal with systems more general than (*S*); thus Hurwicz in no. XVIII considers nonadditive disturbances, and Rubin in no. XIX systems with random coefficients. *H. Wold* (Uppsala).

TOPOLOGY

Müller, Gert H. Über die Eigenschaften der Teilmengen eines Kuratowski'schen Raumes. I. Portugaliae Math. 9, 149–167 (1 plate) (1950).

Let $(R, \bar{\ })$ denote a space satisfying $X \subset Y$ implies $\bar{X} \subset \bar{Y}$, $X \subset \bar{X}$, $\bar{0}=0$, and $\bar{\bar{X}}=\bar{X}$. The author makes an exhaustive analysis of those positional properties (Lagebeziehungen) of subsets of R definable in the form $f(X)Lg(X)$, or $f(X)=0$, or $f(X)=R$. Here f and g may be any monomials in $\bar{\ }$ and "complement" and L is such a relation as inclusion or disjunction. Several quite elaborate tables and diagrams are given. The results are too complex to even sketch. Thus theorem 3 gives 12 necessary and sufficient conditions that R be connected. The last table gives 40 examples realizing in the real number system some of the 64 properties enumerated in table 1.

A. D. Wallace (New Orleans, La.).

Arens, Richard, and Dugundji, J. Remark on the concept of compactness. Portugaliae Math. 9, 141–143 (1950).

Various types of compactness in topological spaces are related. The principal result obtained is: A topological space X has the property that every open covering admits a finite subcovering if and only if every countable open covering admits a finite subcovering and also every open covering admits a point-finite refinement. As an application, it is shown that a paracompact space has the property that every open covering admits a finite subcovering if and only if every countable open covering admits a finite subcovering.

E. Hewitt (Seattle, Wash.).

Norris, Michael J. A note on regular and completely regular topological spaces. Proc. Amer. Math. Soc. 1, 754–755 (1950).

Generalizing a result of the reviewer [Duke Math. J. 10, 309–333 (1943); these Rev. 5, 46], the author proves that the least upper bound of any family of regular (completely regular) topologies on a fixed set is again regular (completely regular).

E. Hewitt (Seattle, Wash.).

Cohen, L. W., and Goffman, Casper. On the metrization of uniform space. Proc. Amer. Math. Soc. 1, 750–753 (1950).

Necessary and sufficient conditions are found for a space with uniform structure to be metrized with distances in an ordered Abelian group.

E. Hewitt (Seattle, Wash.).

Kaluza, Theodor, jr. Struktur- und Mächtigkeitsuntersuchungen an gewissen unendlichen Graphen mit einigen Anwendungen auf lineare Punktmengen. Math. Ann. 122, 235–258 (1950).

See Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig 1947, no. 3; these Rev. 11, 196.

W. T. Tutte (Toronto, Ont.).

Errera, A. Une conséquence d'un théorème de M. Whitney. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 594–596 (1950).

The author obtains a new formulation of the four colour problem by combining P. J. Heawood's formulation in terms of congruences mod 3 with H. Whitney's theorem on Hamiltonian circuits [Ann. of Math. (2) 32, 378–390 (1931)].

W. T. Tutte (Toronto, Ont.).

Finzi, Arrigo. Sur le problème de la génération d'une transformation donnée d'une courbe fermée par une transformation infinitésimale. Ann. Sci. École Norm. Sup. (3) 67, 243–305 (1950).

The parameter x on a closed curve $\mathfrak{C}$ is subjected to a transformation T: $y=g(x)$ having an inverse, having no invariant point, and satisfying $0<g'=dg/dx$. The author asks when can T be imbedded in a one-parameter group, that is, when does g satisfy a differential equation $g'=\xi(g)$? He divides transformations T into three categories: (i) T has finite period; (ii) some power of T leaves some but not all points of $\mathfrak{C}$ invariant; (iii) no power of T leaves a point of $\mathfrak{C}$ invariant, and shows that each T with continuous g' in (i) can be imbedded in any one of an infinity of groups, no T in (ii) can be imbedded in a group, and each T in a subset (iii)′ of (iii) can be imbedded, whereas those in the complementary subset (iii)″ cannot. The nature of a T in (iii) depends upon its modulus, which is defined as the limit for n infinite of m/n, where m is the number of circuits of $\mathfrak{C}$ imposed on x by T^n. The modulus in (iii) is irrational. If the continued fraction for the modulus has quotients a_j and convergents m_j/n_j, a sufficient condition for T in (iii)′ is that $\lim_{j\to\infty} a_j n_j^{-\lambda}=0$ for some λ on $0<\lambda<1$ and g'' satisfy a Lipschitz condition. On the other hand, there exist analytic g's giving T's in (iii)″. A theorem is proved to show that cases (i), (iii)″ are to be regarded as exceptional, whereas an arbitrary transformation is to be regarded as falling under (ii) or (iii)′.

J. M. Thomas.

Magenes, Enrico. Un'osservazione sui teoremi di esistenza di punti uniti in trasformazioni plurivalenti di una N-cella. Rend. Sem. Mat. Univ. Padova 19, 108–113 (1950).

Let I be a closed n-cell and let T be a mapping of I into the closed subsets of I. Conditions which lead to the existence of "fixed points" (i.e., points P such that $P \in T(P)$) have been given by several authors [see, for example, Eilenberg and Montgomery, Amer. J. Math. 68, 214–222 (1946); these Rev. 8, 51]. It is generally assumed that each image set $T(P)$ is connected. The present author considers the possibility of weakening this restriction and shows in particular that the case in which each image has at most two components can, under certain conditions, be reduced to the case of connected images.

P. A. Smith (New York, N. Y.).

Darbo, Gabriele. Grado topologico e teoremi di esistenza di punti uniti per trasformazioni plurivalenti di bicelle. Rend. Sem. Mat. Univ. Padova 19, 371–395 (1950).

Let E be a closed 2-cell in S^2 and let T be a mapping of E into the closed bounded subsets of S^2. Let E_0 be the totality of "fixed-points," i.e., points P such that $P \in T(P)$. Assuming that T satisfies certain conditions, the author defines the degree of T relative to any 1-cycle in $E-E_0$. It is shown that T admits at least one fixed point if its degree relative to the boundary of E (assumed free of fixed points) is different from zero.

P. A. Smith (New York, N. Y.).

*Gerretsen, J. C. H. **Some examples of two-sheeted three-dimensional covering spaces.** Zeven voordrachten over topologie. [Seven Lectures on Topology]. Centrumreeks, no. 1. Math. Centrum Amsterdam, pp. 103–126. J. Noorduijn en Zoon, Gorinchem, 1950. 6 florins. (Dutch)

This paper is partly expository, on two- and three-dimensional manifolds. Four examples are given of two-sheeted branched coverings of the 3-sphere. *W. Kaplan.*

Fet, A. I. On Fomin's conditions for the one-to-oneness of a continuously differentiable transformation. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 163–164 (1950). (Russian)

Fomin [same journal (N.S.) **4**, no. 5 (33), 198–199 (1949); these Rev. **11**, 195] confines himself to the case of a convex domain, suggesting, however, that the limitation of convexity be removed. The present note shows by a simple example that the Fomin result does not hold for nonconvex domains. *L. Zippin* (Brooklyn, N. Y.).

Dehn, Max. Über Abbildungen geschlossener Flächen auf sich. Mat. Tidsskr. B. **1950**, 146–151 (1950).

Let Γ_p be the group of continuous orientation-preserving automorphisms of a closed surface of genus p, modulo the identity-homotopic automorphisms. On representing Γ_p by relations on certain generators S_i, the author considers the effect of adjoining further relations. The relations $S_iS_jS_i^{-1}S_j^{-1}=1$ reduce Γ_p to a finite cyclic group. For $p=2$, the relations $S_i^3=1$ reduce Γ_p to the symmetric group on six letters. *P. A. Smith* (New York, N. Y.).

Liao, S. D. Concerning the generators of homotopy groups of a polyhedron. Proc. Amer. Math. Soc. **1**, 763–768 (1950).

The author proves the following influence of π_1 on π_n: Let P be a connected finite n-polytope, $n\geqq 2$. If $\pi_i(P)=0$ for $1<i<n$, and $\pi_n(P)\neq 0$, then $\pi_n(P)$ has a finite number of generators if and only if $\pi_1(P)$ is a finite group. The following result on identifications is also proved: If P is a connected polytope, and P' the polytope obtained by identifying two distinct points of P, then whenever $n\geqq 2$ and $\pi_i(P)=0$, $1<i<n$, $\pi_n(P')$ is the weak direct sum of a countably infinite collection of groups, each isomorphic to $\pi_n(P)$. These two results are proved by the standard technique of passage to the universal covering, and use of the Hurewicz isomorphism theorem therein.

J. Dugundji (Los Angeles, Calif.).

Blakers, A. L., and Massey, W. S. The homotopy groups of a triad. I. Ann. of Math. (2) **53**, 161–205 (1951).

A detailed exposition of most of the results announced in a previous note [Proc. Nat. Acad. Sci. U. S. A. **35**, 322–328 (1949); these Rev. **11**, 47].

J. Dugundji (Los Angeles, Calif.).

Shizuma, Ryoji. Fibre bundles over spheres. Nagoya Math. J. **1**, 59–61 (1950).

Let R be a fibre bundle with fibre F and base space an n-sphere. Assume that the fibre is arcwise connected and simple in dimension $n-1$. The author proves a theorem which tells how the homotopy group $\pi_{n-1}(R)$ is determined by $\pi_{n-1}(F)$ and the characteristic cohomology class of the bundle. As an application of this theorem, he reproves a theorem of Stiefel [Comment. Math. Helv. **8**, 305–353 (1936)] about the homology groups of the Stiefel manifolds, $V(n, r)$. *W. S. Massey* (Providence, R. I.).

Borel, Armand. Impossibilité de fibrer une sphère par un produit de sphères. C. R. Acad. Sci. Paris **231**, 943–945 (1950).

The author announces and indicates the proofs of some results on the cohomology structure of fiber bundles. If the fiber F has its cohomology ring isomorphic with that of a product of s spheres of odd dimensions m_i, and the bundle E has trivial cohomology up to dimension $n\geqq\sum m_i$, then the base B has its cohomology ring up to dimension n isomorphic with the ring of polynomials of s variables q_i with degrees m_{i+1}. [Field coefficients are used; if the characteristic is not 2, the m_i can be proved to be odd.] If the fiber is as above (but m_i even allowed, if the characteristic $\neq 2$) and the ring of E is isomorphic with the ring of the sphere S^n, then $s=1$ and m_1 odd, B is assumed finite dimensional. If Euclidean space R^n is fibered (locally trivial) with connected fiber F, and base B, then F and B have compact cohomology groups isomorphic with those of R^p and R^q ($p+q=n$). The proofs utilize Leray's spectral ring, for the projection of E into B. *H. Samelson.*

Hu, Sze-Tsen. On generalising the notion of fibre spaces to include the fibre bundles. Proc. Amer. Math. Soc. **1**, 756–762 (1950).

The author gives a new definition of fibre space which is more general than that originally given by Hurewicz and Steenrod [Proc. Nat. Acad. Sci. U. S. A. **27**, 60–64 (1941); these Rev. **2**, 323], and shows that with his new definition, the covering homotopy theorem is still true. The new definition has the advantage that it is now possible to prove that every fibre bundle is a fibre space; it is still unknown whether or not every fibre bundle is a fibre space in the sense of Hurewicz and Steenrod.

W. S. Massey (Providence, R. I.).

Hu, Sze-tsen. Homology groups of a ring. I. Bull. Calcutta Math. Soc. **42**, 123–130 (1950).

Homology groups are defined for an abstract ring C, which, when C is the ring of all continuous real functions over a compact metric space S, are isomorphic to the Alexander-Kolmogoroff-Lefschetz groups for S. The definition is in terms of maximal ideals, otherwise making little use of the ring operations, and is such that the correspondence from a function to its null-space induces an isomorphism of the chain groups. *R. C. Lyndon.*

Ma, Liang. On relations between homology and homotopy groups. Portugaliae Math. **8**, 107–135 (1949).

Il s'agit essentiellement d'une revue des méthodes et de quelques résultats classiques d'Eilenberg et MacLane, concernant l'application de la cohomologie des groupes discrets à l'étude des relations entre la cohomologie d'un espace et le groupe fondamentale de cet espace [l'auteur ne cite pas le mémoire d'Eckmann, Comment. Math. Helv. **18**, 232–282 (1946); ces Rev. **8**, 166]. Le théorème fondamental est un peu amélioré, par léger affaiblissement des hypothèses. Quelques compatibilités d'homomorphismes sont démontrées.

H. Cartan (Paris).

GEOMETRY

Carlebur, F., and Schuh, Fred. An approximate construction of the trisectors of an angle and an improvement of it. Nieuw Tijdschr. Wiskunde **38**, 164–173 (1951). (Dutch)

Goormaghtigh, R. Sur les groupes de triangles. Mathesis **59**, 304–307 (1950).

Ramler, O. J. A poristic system of triangles. Duke Math. J. **17**, 443–451 (1950).

The author considers Poncelet [Traité des propriétés projectives . . . , v. 1, 2d ed., Gauthier-Villars, Paris, 1865, article 565] systems of triangles inscribed in a conic (circle) and circumscribed about another conic (circle). Making skillful use of isotropic coordinates he proves a considerable number of propositions, most of which have been obtained originally by such writers as Laguerre, M. Weill, and more recently by Weaver, Goormaghtigh, and others. The paper concludes with an adequate list of bibliographical references. *N. A. Court* (Norman, Okla.).

Bruins, E. M. Sur la droite d'Euler. Nieuw Tijdschr. Wiskunde **38**, 152–156 (1951).

Thébault, Victor. Sur les points de Steiner et de Tarry. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 131–138 (1950).

Thébault, Victor. On a theorem of Steiner. Amer. Math. Monthly **58**, 25–27 (1951).

Thébault, Victor. A curve associated with a tetrahedron. Amer. Math. Monthly **57**, 663–667 (1950).

Leech, John. Some properties of the isosceles tetrahedron. Math. Gaz. **34**, 269–271 (1950).

van Beylen, E. The regular star-shaped polygons. Nieuw Tijdschr. Wiskunde **38**, 145–151 (1951). (Dutch)

Hugon, P. Sur une représentation rectiligne des hyperboles sphériques. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. **38**, v+19 pp. (1950).

Loeffler, A. Construction du cercle osculateur en un point quelconque d'une quartique bicirculaire. Elemente der Math. **6**, 11–15 (1951).

Lorent, H. Une transformation de lignes et de surfaces. Anais Fac. Ci. Porto **33**, 129–154 (1948).

Lorent, H. Transformations de courbes planes. IV. Anais Fac. Ci. Porto **33**, 82–95 (1948).

For the first three parts cf. same Anais **26**, 5–20, 65–83 (1941); **36**, 129–144 (1946); these Rev. **9**, 301; **10**, 619.

***Bieberbach, L. Einführung in die analytische Geometrie.** 4th ed. Verlag für Wissenschaft und Fachbuch, Bielefeld, 1950. 168 pp. 8.90 DM.

Locher-Ernst, L. Polarentheorie der Eilinien. Elemente der Math. **6**, 1–7 (1951).

Ballieu, Robert. Sur les triangles autopolaires des polarités planes. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 48–54 (1950).

The author considers self polar triangles of polarities in the projective plane over an arbitrary ground field. A uniform polarity is one with no self-conjugate points. If a polarity is nonuniform a self-polar triangle might have 0, 1, 2, or 3 of its sides containing self-conjugate points, and for a given polarity this number need not be the same for all self-polar triangles. The author classifies all the possible cases which may arise in terms of four simple conditions on the ground field (all concerning the elements of the field which may be squares), the various cases corresponding to different combinations of the four conditions and their negatives. *D. B. Scott* (London).

Blumenthal, Leonard M. Generalized Euclidean space in terms of a quasi inner product. Amer. J. Math. **72**, 686–698 (1950).

Definitions and notations. Let Σ denote an abstract set. To each pair of elements x, y of Σ is attached a real number (x, y), called quasi inner product. For n elements $x_1, \cdots, x_n$ of Σ, $G(x_1, \cdots, x_n)$ represents the Gram determinant $|(x_i, x_j)|$, $i, j = 1, 2, \cdots, n$, $B(x_1, \cdots, x_n)$ the symmetric determinant obtained by bordering $G(x_1, \cdots, x_n)$ with a row and column of 1's with intersection element 0. *Ex*: There exists at least one x. (x): For each x. (λ): For each real number λ. Postulates. (Q_1) $(x, y) = (y, x)$; (Q_2) $(x, x) \geqq 0$; (Q_3) $((x, x) = (x, y) = (y, y)) \rightarrow (x = y)$; (S_1) $G(x, y) \geqq 0$; (S_2) $(G(x, y) = 0) \rightarrow (G(x, y, z) \geqq 0)$; (S_3) $(G(x, y, z) = 0) \rightarrow G(x, y, z, t) \geqq 0)$; (E_1) $(E\theta)(x)((\theta, x) = 0)$; (E_2) $(x)(\lambda)(Ey)((G(x, y) = 0)$ and $((x, y) = \lambda(x, x))$; (E_3) $(G(x, z) \neq 0) \rightarrow (Ey)((B(x, y, z) = G(x, y, z) = 0)$ and $(G(x, y) = G(y, z))$. Main theorem. If the above listed postulates are satisfied, Σ is a generalized Euclidean space. Sketch of the proof. For $x \neq \theta$, $y = \lambda \cdot x$ is defined by the two properties: $(x, y) = \lambda(x, x)$ and $G(x, y) = 0$. It follows that y is a scalar multiple of x if and only if $G(x, y) = 0$. To each two elements x, y is attached the number $xy = [(x, x) + (y, y) - 2(x, y)]^{\frac{1}{2}}$ as distance. The norm $\|x\|$ of x is defined as $x\theta$. If $x \neq z$, an element y of Σ is an algebraic middle element of x and z provided (a) $G(x, z) \neq 0$, x, y, z satisfy $B(x, y, z) = G(x, y, z) = 0$ and $G(x, y) = G(y, z)$ or (b) $G(x, z) = 0$ and $y = [(1 + \lambda)/2] \cdot x$, where $z = \lambda \cdot x$ (with the roles of x and z reversed in the two immediately preceding equalities in case $x = \theta$). The sum $x + z$ is defined as 2(algebraic middle element). Then $\lambda \cdot x$ and $x + z$ exist and are unique. The axioms for a normed linear space are fulfilled. The function (x, y) is an inner product and $\|x\| = (x, x)^{\frac{1}{2}}$. The concluding section of the paper is concerned with the congruent and vectorial application of Σ onto Euclidean and Hilbert spaces, yielding the following metric characterizations: Σ is congruent with Hilbert space provided it is separable, complete, and for each positive integer k a subset $x_1, \cdots, x_k$ of Σ exists with $G(x_1, \cdots, x_k) \neq 0$. The set Σ is congruent with the n-dimensional Euclidean space provided it is complete, and n is the smallest positive integer such that each $n+1$ elements of Σ have a vanishing determinant G. *C. Y. Pauc* (Cape Town).

Christov, Chr. Sur les distances entre les points d'un espace euclidien ou pseudo-euclidien. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **45**, 439–466 (1949). (Bulgarian. French summary)

Let (a_{ij}) denote any given set of real numbers with $a_{ij} = a_{ji}$ and $a_{ii} = 0$, $i, j = 0, 1, \cdots, n$. The object of this paper is to show that there exist (1) $n+1$ ordered n-tuples of real numbers $(x_1^{(0)}, \cdots, x_n^{(0)})$, $i = 0, 1, \cdots, n$, and (2) real numbers (c_{ij}) with $c_{ij} = c_{ji}$, $i, j = 1, \cdots, n$, such that $a_{ij} = \sum_{\alpha, \beta=1}^{n} c_{\alpha\beta}(x_\alpha^{(i)} - x_\alpha^{(j)})(x_\beta^{(i)} - x_\beta^{(j)})$, $i, j = 0, 1, \cdots, n$. A

stronger form of this theorem (in which $c_{ij}=0$ $(i\neq j)$, $c_{ii}=1$ $(1\leqq i\leqq\frac{1}{2}(n+s))$, $c_{ii}=-1$ $(\frac{1}{2}(n+s)<i\leqq n)$, and s is determined in a definite manner from the given set of numbers) was established by Wald [Ergebn. Math. Kolloqu. Wien 5, 32–42 (1933); especially pp. 36–40] and a similar result was obtained by Coxeter and Todd [Proc. Cambridge Philos. Soc. 30, 1–3 (1934)]. The author's proof is quite lengthy.

L. M. Blumenthal (Columbia, Mo.).

Convex Domains, Extremal Problems

Peck, L. G. On cube-coverings of n-space. Amer. J. Math. 72, 720–722 (1950).

Hajós [Math. Z. 47, 427–467 (1941); these Rev. 3, 302] has shown that in every covering of Euclidean n-space by unit cubes, whose edges are parallel to the coordinate axes and whose centers form a lattice, there is a pair of cubes with a complete $(n-1)$-dimensional face in common. The present paper contains the results of an attempt to generalize this theorem by omitting the condition that the centers of the cubes form a lattice. A subset $\mathfrak{A}$ of the n-dimensional vector space $\mathfrak{R}_n$ over the real field is called an n-grill if, for every $\mathfrak{x}\in\mathfrak{R}_n$, there exists a unique $\mathfrak{a}\in\mathfrak{A}$ such that $0\leqq x_i-a_i<1$ $(i=1, 2, \cdots, n)$. With every n-grill one can associate a cube-covering and vice versa. It is shown that in any cube-covering of this type each cube has a vertex in common with some other cube; i.e., if $\mathfrak{p}\in\mathfrak{A}$ there exist numbers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ not all zero, such that $\epsilon_i=0$ or 1 $(i=1, 2, \cdots, n)$ and $\mathfrak{p}+\epsilon_1\mathfrak{e}_1+\epsilon_2\mathfrak{e}_2+\cdots+\epsilon_n\mathfrak{e}_n$, where $\mathfrak{e}_1, \mathfrak{e}_2, \cdots, \mathfrak{e}_n$ are the unit vectors parallel to the axes. Misprint: In the proof of the lemma read $-x_i+1$ for $-x_i-1$ throughout in the case $\delta_i=+1$.

R. A. Rankin (Cambridge, England).

Bouligand, Georges. Sur les figures polygonales convexes du plan et de la sphère. Rev. Gén. Sci. Pures Appl. N.S. 57, 156–160 (1950).

***Weyl, Hermann. Elementary proof of a minimax theorem due to von Neumann.** Contributions to the Theory of Games, pp. 19–25. Annals of Mathematics Studies, no. 24. Princeton University Press, Princeton, N. J., 1950. $3.00.

The following quotation summarizes the paper: "J. von Neumann's minimax problem in the theory of games belongs to the theory of linear inequalities and can be approached in the same elementary way in which I proved the fundamental facts about convex pyramids. As elementary are considered such operations in an ordered field K of numbers as require nothing but addition, subtraction, multiplication, and division, and the decision whether a given number is >0 or $=0$ or <0. Decisions about a set of numbers are elementary only if they concern a finite set, the members of which are exhibited one by one. In such a sequence of numbers $\alpha_1, \cdots, \alpha_n$ we can find the smallest, min α_k, and the biggest, max α_k. As for the field K no continuity axioms, not even the axiom of Archimedes, are assumed."

J. Wolfowitz (New York, N. Y.).

Valentine, F. A. Some properties of C-convex sets. Canadian J. Math. 2, 481–488 (1950).

The main result of the present paper can be stated as follows: Each pair of points of the two-dimensional complement C of $n\geqq 1$ nonoverlapping closed convex domains can be joined by a polygonal arc in C having at most $n+1$ segments.

L. Fejes Tóth (Veszprém).

Bieri, H. Die 1. Variation der Masszahlen eines Elementarkegels. Comment. Math. Helv. 24, 232–237 (1950).

A useful approach to many problems in the theories of surfaces and convex bodies is the use of a variational technique. The author computes the variations of a number of quantities listed below. Consider an elementary portion W of the surface of a convex body, the curve c bounding W, a strip S containing c, and the cone formed by joining W to an interior point of the body. Deformations of the following form are permitted:

$$\bar{\mathfrak{x}}=\mathfrak{x}+\epsilon\{p(u, v)\mathfrak{x}_u+q(u, v)\mathfrak{x}_v+n(u, v)\mathfrak{N}\},$$

where $\mathfrak{x}$ is the position vector of W, $\mathfrak{x}_u$, and $\mathfrak{x}_v$ tangent vectors to W, and $\mathfrak{N}$ the unit inner normal. Let the variation be defined as $\delta=\epsilon(\partial/\partial\epsilon)_{\epsilon=0}$. Then the variations of the following are computed: (1) F, the surface area of W; (2) V, the volume of the cone; (3) M_1, the integral of the mean curvature of W; (4) M_2, the integral around c of the angle between $\mathfrak{N}$ and the normal to S (where S undergoes a deformation of the same type that applied to W); (5) s, the length of c; and (6) ω, the solid angle at the vertex of the cone.

C. B. Allendoerfer (Haverford, Pa.).

***Pogorelov, A. V. The rigidity of convex surfaces.** Trudy Mat. Inst. Steklov 29, 99 pp. (1949). (Russian)

A surface S in E^3 is said to have bounded partial curvature if a number M exists such that the ratio of the measure of the spherical image of a domain on S to the measure of the domain itself stays below M (all concepts are to be taken in the general sense defined by A. D. Aleksandrov). If F_1 is a closed convex surface of bounded partial curvature and F_2 is any convex surface (intrinsically) isometric to F_1, then F_2 is congruent to F_1 (that is, can be carried into F_1 by motions, including reflections). Since in this result no assumption is made regarding F_2 beyond the convexity, it gives the first geometrically completely satisfactory answer to a type of problem first treated by Cohn-Vossen. For instance, it contains the result that a convex surface isometric to an ellipsoid is an ellipsoid. Another consequence is that a closed convex surface whose intrinsic metric is analytic and has positive Gauss-curvature is analytic as a surface in E^3. A convex cap is a convex surface S bounded by a plane curve C but such that the projection of S on the plane of C falls inside C. If F_1 is a convex cap of bounded partial curvature and F_2 is a convex cap isometric to F_1, then F_2 is congruent to F_1. Let F_1 be a convex surface which is complete in the sense of the inner metric, z a fixed point of F_1, k_r the circular disk on F_1 about z with (geodesic) radius r, $l(r)$ the length of the shortest curve on F_1 containing k_r, and $\varphi(r)$ the total curvature of k_r. If $[2\pi-\varphi(r)]/l(r)\to 0$ for $r\to\infty$, then any convex surface F_2 isometric to F_1 is congruent to F_1. The theorem remains true if the condition of completeness is removed and F_1 is permitted to have a finite number of holes, provided the total geodesic curvature (in the sense of Aleksandrov) of each boundary curve is nonpositive. There are also results for complete surfaces with total curvature less than 2π. Since such surfaces are, in general, not congruent when they are isometric, additional hypotheses become necessary, which are too involved to be formulated here. They are in the spirit of those obtained by Olovjanishnikov [Rec. Math. [Mat. Sbornik] N.S. 18(60), 429–440 (1946); these Rev. 8, 169]. The article

is very readable since pp. 11–19 are dedicated to a detailed explanation of the methods used in the proofs.

H. Busemann (Los Angeles, Calif.).

Pinl, Max. Über die komplexen Minimalflächen mit der Gaussschen Krümmung Null. Arch. Math. **2**, 283–288 (1950).

Minimal surfaces in Euclidean n-space are conveniently represented in isotropic parameters and are translation surfaces of isotropic curves. Introducing such parameters, the author considers the minimal surfaces with vanishing Gaussian curvature. The types of such surfaces are classified for the cases $n=3, 4$. It is stated that for general n, there are $\frac{1}{2}(n+1)(n-2)$ cases to be considered.

S. B. Jackson (College Park, Md.).

Algebraic Geometry

Godeaux, Lucien. Sur la jacobienne d'un réseau de courbes algébriques. Mathesis **59**, 297–299 (1950).

Galafassi, Vittorio Emanuele. Indirizzi e metodi in "questioni di realità." Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 77–93 (1950).

Lecture given in May, 1950.

Burniat, Pol. Sur les surfaces canoniques quadruples. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 1602, 31 pp. (1950).

L'auteur indique, dans ce travail, un mode simple de construction de surfaces F, birationnellement identiques aux surfaces canoniques quadruples dont l'involution d'ordre 4 est composée avec une involution d'ordre 2, régulière ou irrégulière. Il montre en particulier que ces surfaces F sont des cônes cubiques elliptiques ou des surfaces rationnelles, et que dans les deux cas elles existent pour toutes les valeurs du genre et de l'irrégularité. L'auteur construit effectivement toutes ces surfaces. *M. Piazzolla-Beloch.*

Roth, Leonard. Su alcune varietà algebriche che contengono un sistema di curve ellittiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 62–64 (1950).

Castelnuovo has proved [same Rend. Cl. Sci. Fis. Mat. Nat. (5) 3_1, 473–481 (1894)] that every surface which carries a net of elliptic curves is rational or birationally equivalent to an elliptic cone. In the present note the author outlines an extension of Castelnuovo's methods and results to the case of a V_r which carries an $r+k-1$-dimensional ($k \leqq r$) system Σ of elliptic curves C such that through the general point P of V_r there passes an irreducible subsystem $\{C\}$ of Σ of dimension k. The cases $k=r$ and $2k \geqq r$ are considered in particular, and various applications of the results are given. *H. T. Muhly* (Iowa City, Iowa).

Baldassarri, Mario. Le varietà pluririgate a tre dimensioni. Rend. Sem. Mat. Univ. Padova **19**, 172–200, 340–341 (1950).

Considérant une surface F de S^3, le complexe K des tangentes se représente par une variété K^* située sur la quadrique Q de Klein, image des droites de S^3, chaque droite de cette K^* a deux foyers, la surface H^* image de la congruence des bitangentes à F est double pour K^*, la courbe T^* image de la réglée des tritangentes est triple pour H^* et K^*; ces singularités normales dans S^5 étant seules satisfaites on dira le complexe K régulier; les singularités d'ordre supérieur ne se présentent que si K^* contient des plans multiples ou des courbes multiples dont les tangentes appartiennent à Q. Une variété R^3 possédant ∞^2 droites dans S^4 possède $n-3$ points doubles sur chaque génératrice; les plans passant par celles-ci sont bitangents et sont en projectivité avec les couples de points de la génératrice. Une R^3 est pluriréglée si par un point générique passent γ droites; les droites s'appuyant à l'une d'elles forment une réglée dont elle est directrice d'ordre $\gamma-1$; dans cette note l'auteur précise exclure les R^3 contenant un systéme algébrique ∞^1 de quadriques. Le lieu des points doubles de la R^3 est une surface d'ordre $\geqq 3(n-3)-1$. Il apparait immédiatement que si $n \leqq 6$, les pluriréglées sont les variétés à courbes sections elliptiques normales de S^{n+1}, excepté la R_3^4 à droite double de S^7. Considérant le grassmanienne M_6^5 des droites de S^4, et remarquant que l'indice ne peut dépasser 6 (appartenance aux trois premiéres polaires), les réglées immergées dans R^3 forment sur M_6^5 un systéme de courbes dont le degré et l'indice égaux ne peuvent prendre que les valeurs 1 et 2 d'où l'on conclut en construisant dans M l'image des droites de R_3 que $n<7$. Les seuls cas sont donc ceux trouvés supra. Se lie à cette étude la liste des surfaces de la M coupée selon des courbes dont les S^3 sont tangents à la M en ces points. *B. d'Orgeval* (Grenoble).

Tanturri, Giuseppe. Su certi inviluppi proiettivamente legati ai fasci di curve piane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 213–218 (1950).

Für die allgemeine Kurve eines Büschels sei ein bestimmter Basispunkt O kein Wendepunkt. Es seien die Kegelschnitte ins Auge gefasst, die ein festes Linienelement enthalten und die Kurven des Büschels in O dreipunktig berühren. Verf. bestimmt die Enveloppen dieser Kegelschnittscharen. Das Ergebnis hat verschiedene Gestalt, je nachdem die drei Kurven des Büschels, die in O einen Wendepunkt haben, verschieden sind oder zu zweien oder zu dreien zusammenfallen. Verf. betrachtet weiter Paare von Büscheln durch O, für die jene drei Kurven sich paarweise berühren. Dann haben alle Paare sich in O gegenseitig berührender Kurven dort eine konstante Mehmke-Segresche Kontaktinvariante, und die beiden Kegelschnittscharen sind projektiv. Verf. verallgemeinert seine Ergebnisse noch auf den Fall, dass die allgemeine Kurve in O eine Wendetangente, allgemeiner eine p-fache Tangente besitzt. *O.-H. Keller* (Dresden).

Villa, Mario, e Vaona, Guido. Alcune osservazioni sulle curve caratteristiche delle trasformazioni cremoniane. Boll. Un. Mat. Ital. (3) **5**, 101–107 (1950).

Bei einer ebenen Cremona-Transformation haben von den Bildern der Geraden, die durch einen bestimmten Punkte gehen, drei dort eine Wendetangente. Die Richtungen derselben heissen charakteristische Richtungen. Durch sie ist eine Differentialgleichung 1. Ordnung 3. Grades gegeben; ihre Integrale sind die charakteristischen Kurven der Transformation. Verf. gibt ein Beispiel einer kubischen Transformation, deren charakteristische Kurven transzendent sind. Jeder Bestandteil der Jacobischen Kurve der Transformation ist ein singuläres Integral der charakteristischen Differentialgleichung und gehört ihrer Diskriminantenkurve an, es sei denn, dass er ein mehrfacher Bestandteil ist und die charakteristischen Richtungen in ihrem allgemeinen Punkt unbestimmt werden. *O.-H. Keller* (Dresden).

van der Kulk, W. The univalent algebraic transformations of the projective plane. Duke Math. J. 17, 367–383 (1950).

A univalent transformation of the projective plane P is an irreducible transformation T of P into itself such that, for every point s of P which is algebraic over a field of definition Δ of T, T_s and T_s^{-1} are uniquely defined. It is well known that in the case in which Δ is the field of complex numbers every univalent transformation of P is a projective transformation. A proof is first given that this theorem remains true if Δ is any field of characteristic zero. If Δ is of characteristic $p>0$, the theorem is not true. The case in which Δ is a perfect field is examined in detail. It is shown that the group of all univalent transformations of P is generated by the semi-projective transformations. A semi-projective transformation is one which, for a suitable choice of triangle of reference, can be written as $y_0=[x_0^{p^m}+\sigma(x_1, x_2)]^{p^{m-e}}$, $y_1=x_1^{p^{m-e}}$, $y_2=x_2^{p^{m-e}}$, where e is an integer, and $\sigma(x_1, x_2)$ is a homogeneous polynomial of degree p^m which is not the pth power of another polynomial. If $m\geqq 1$, these equations define a proper semi-projective transformation, and if $m=0$ they define a p-projective transformation. Any univalent transformation T, not a p-projective transformation, can be expressed as a product of a finite number of proper p-transformations, and if $S_1, \cdots, S_n$ and $S_1', \cdots, S_{n'}'$ are two minimal representations of T, $n=n'$, and $S_i=T_i^{-1}S_i'T_{i+1}$, where $T_1=T_{n+1}=1$, and T_i is a p-projective transformation ($i=2, \cdots, n$). The results can be extended to the case in which Δ is not a perfect field, by giving a suitable definition of a semi-projective transformation.

W. V. D. Hodge (Cambridge, England).

Martinelli, E. Geometria algebrica e geometria riemanniana. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 1–25 (1950).

An expository account of the relations between Riemannian and algebraic geometry, dealing with the properties of the Kähler metric on an algebraic variety, harmonic integrals, and the applications of the theory of harmonic integrals to problems of algebraic geometry.

W. V. D. Hodge (Cambridge, England).

Differential Geometry

Löbell, Frank. Bemerkungen zum Beweise des Gauss-Bonnetschen Satzes. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1949, 21–35 (1950).

The Gauss-Bonnet theorem is proved for a multiply-connected region on a surface. The proof assumes that the surface lies in Euclidean space and uses the spherical representation of Gauss. The theorem is first proved for special 2-gons and then for special triangles. The given region is then triangulated so that only these special triangles appear, and the result follows by summation. The usual statements of the theorem require the surface to be orientable, but the author shows that on a Möbius strip the scalar form of the theorem is trivially true and the vector form remains true without change.

C. B. Allendoerfer (Haverford, Pa.).

Terracini, Alessandro. Congruenze *W*. Matematiche, Catania 5, 83–90 (1950).

Expository lecture.

Behari, Ram, and Mishra, Ratan Shanker. Bianchi's characteristic function in a congruence of Ribaucour. Proc. Nat. Inst. Sci. India 16, 83–86 (1950).

Les congruences de Ribaucour sont formées par les parallèles menées des points d'une surface (de référence) aux normales d'une autre surface, (directrice) correspondant à la première avec orthogonalité des éléments linéaires. La surface de référence étant donnée, Weingarten a ramené la détermination de la surface directrice (donc de la congruence de Ribaucour) à celle d'une certaine fonction Φ, dite par Bianchi fonction caractéristique. Les auteurs donnent une expression de Φ au moyen des éléments infinitésimaux de la surface de référence et de la surface directrice. Si θ est l'angle d'un rayon de la congruence de Ribaucour avec la normale à la surface de référence, et si ds_1 et ds sont les éléments d'aire des deux surfaces, on a $\Phi^2=\cos\theta\, ds_1/ds$, et l'on en déduit $\Phi^2=-\kappa\rho^2$, où κ est la courbure totale de la surface directrice et ρ la distance commune de deux foyers associés de la congruence au point correspondant de la surface de référence. L'angle θ est susceptible d'expressions simples au moyen des tenseurs fondamentaux de la surface de référence et de la surface directrice. Les auteurs en déduisent quelques conséquences plus ou moins immédiates, et en particulier la suivante: $\Phi\Phi_1=\cos\theta$, où Φ_1 est la fonction caractéristique obtenue en échangeant les rôles de la surface de référence et de la surface directrice.

P. Vincensini (Marseille).

***Unger, Georg. Krümmungsfeste und wackelige Kurvennetze bei infinitesimalen Verbiegungen als Analoga zu den Asymptotenlinien und konjugierten Netzen.** Thesis, University of Zürich, 1941. 65 pp.

Given a parametric surface $\mathfrak{x}=\mathfrak{x}(u, v)$ with the first and second fundamental forms I and II. Suppose the surface $\mathfrak{x}_1=\mathfrak{x}+\epsilon\bar{\mathfrak{x}}$ is obtained by means of an infinitesimal deformation $\mathfrak{D}$ of $\mathfrak{x}$. Thus $\mathfrak{x}_1$ has the first fundamental form $\mathrm{I}+\epsilon^2\cdot *$, while its second fundamental form is equal to $\mathrm{II}+\epsilon\cdot\mathrm{V}+\epsilon^2\cdot *$, where V is the first variation of II. The author studies $\mathfrak{D}$ by means of the "deformation form" V. In this way, some of the deformation properties of a surface appear as close analogues of its curvature properties. The following results may be of interest. At each point of $\mathfrak{x}$ there are two, necessarily conjugate, directions for which V vanishes. They determine the system of those lines for which the first variation of the normal curvature of $\mathfrak{x}$ is zero ["krümmungsfeste Linien"]. It corresponds to a system of the same type on the associated surface $\mathfrak{y}$ and to a system of asymptotic lines on $\bar{\mathfrak{x}}$. Two directions are $\mathfrak{D}$-conjugate if the polar form of V vanishes for them. They correspond to conjugate directions on $\bar{\mathfrak{x}}$, and $\mathfrak{D}$-conjugate systems are "wackelig" in the sense of R. Sauer [Math. Ann. 108, 673–693 (1933)]. The two "principal $\mathfrak{D}$-directions" correspond to the extremums R_1^{*-1} and R_2^{*-1} of V/I. They determine the real, orthogonal and $\mathfrak{D}$-conjugate system of principal $\mathfrak{D}$-lines. The first variation of the geodesic torsion of these lines vanishes. The principal $\mathfrak{D}$-lines of $\mathfrak{x}$ and $\mathfrak{y}$ correspond to each other. The sum $R_1^{*-1}+R_2^{*-1}$ is the first variation of the mean curvature. The $\mathfrak{D}$-measure $(R_1^*R_2^*)^{-1}$ of $\mathfrak{x}$ is equal to the quotient of the Gauss curvatures of $\mathfrak{x}$ and of $\mathfrak{y}$, while the total $\mathfrak{D}$-measure $\int(R_1^*R_2^*)^{-1}do$ of some part of the surface $\mathfrak{x}$ is equal to the area of the corresponding part of the surface $\mathfrak{y}$.

P. Scherk (Saskatoon, Sask.).

***Rubin, Frédéric. Zur Deformation singularitätenfreier Flächen.** Thesis, University of Zürich, 1950. 45 pp.

[Cf. the preceding review for terminology.] The surfaces $\mathfrak{x}$ and $\mathfrak{x}_1$ are assumed to have no singularity. More accurately, to each point of $\mathfrak{x}$, respectively $\mathfrak{x}_1$, there is a coordinate system in which $\mathfrak{x}$ ($\mathfrak{x}_1$) has the following canonical expansion about that point: $z=ax^2+by^2+$higher terms; $a^2+b^2>0$. The author studies the canonical expansions of the surfaces $\bar{\mathfrak{x}}$ and $\mathfrak{y}$ and discusses the shape of an image on $\mathfrak{y}$ of a neighborhood on $\mathfrak{x}$. This shape depends essentially on the behavior of the $\mathfrak{D}$-measure of $\mathfrak{x}$ in this neighborhood. In the last section, $\mathfrak{x}$ is assumed to be a closed surface of genus 0. Suppose $\mathfrak{y}$ is closed. By means of the above discussion, it is proved that the area of those parts of the spherical image of $\mathfrak{y}$ that correspond to points of positive Gauss curvature of $\mathfrak{y}$ exceeds the area of those parts that correspond to points of negative curvature. Since this readily leads to a contradiction, $\mathfrak{y}$ cannot be a closed surface. Hence, $\mathfrak{y}$ must degenerate to a point, i.e., $\mathfrak{D}$ is a Euclidean motion. Thus $\mathfrak{x}$ is rigid under this class of infinitesimal deformations $\mathfrak{D}$ [parts of this section are not quite clear to the reviewer].

P. Scherk (Saskatoon, Sask.).

Van Bouchout, V. A theorem on isometric lines and isometric parameters. Simon Stevin **27**, 133–135 (1950). (Dutch)

This paper gives a proof of the theorem that the isogonal trajectories of a system of isothermal lines on a surface are also isothermal. The demonstration shows how to find the equation $f(u, v)=c$, c a constant, of the isogonal isothermic lines of the given family $\varphi(u, v)=0$ in such a way that when φ is an isothermic parameter f is also an isothermic parameter [the theorem is stated by Ricci, Lezioni sulla teoria della superficie, Drucker, Verona-Padova, 1898, article 88].

D. J. Struik (Cambridge, Mass.).

Simonart, Fernand. Sur les configurations hexagonales. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 454–460 (1950).

The main result of this second part [for the first part see the same vol., 268–284 (1950); these Rev. **12**, 128] may be stated as follows: Let the 3-web be defined by $x=x_0$, $y=y_0$, $F(x, y)=F(x_0, y_0)$; suppose $F(0, 0)=0$ and put $F(x, 0)=X$, $F(0, y)=Y$, $F_1(X, Y)=F(x, y)$. Then the last side of the heptagon which one obtains in the usual way in a 3-web is of length $\epsilon'=F_1(Y, X)-F_1(X, Y)$. This formula leads by means of an easy straightforward computation to the Thomsen result. The author applies it on the case of isothermic curves.

V. Hlavatý (Bloomington, Ind.).

Grove, V. G. A note on isothermal nets. Proc. Amer. Math. Soc. **1**, 595–599 (1950).

Let the differential equation of a one-parameter family of curves C_λ on a surface S be $M\,du-N\,dv=0$, $\lambda=M/N$, referred to asymptotic parameters u, v. Let y be a point on the tangent to the curve C_λ through a generic point x of S, but not on the R-harmonic line $(x_u x_v)$. [For the definition of this line and of the R-conjugate line defined by the author see Bull. Amer. Math. Soc. **36**, 582–586 (1930), p. 584.] The projections r, s of y from the points x_v and x_u on the asymptotic tangents are defined by $r=x_v+Sx/N$, $s=x_v+Sx/M$. The author has called these points the integral generating points [Duke Math. J. **15**, 95–103 (1948); these Rev. **9**, 531]; in this paper, however, it is not required that they be invariant. The tangent to the locus of $s(r)$ as x describes the curve $v=$a constant ($u=$a constant) intersects the plane determined by the tangent to $u=$a constant ($v=$a constant) and the R-conjugate line in a point whose projection from x_{uv} on the tangent plane is the point $\rho(\sigma)$ given by

$$\rho=x_u+\left(\frac{\partial}{\partial u}\log\frac{S}{M}\right)x,\quad \sigma=x_v+\left(\frac{\partial}{\partial v}\log\frac{S}{N}\right)x.$$

Let points ρ_h, σ_h be defined by the property that one of the cross-ratios of x, x_u, ρ, ρ_h and of x, x_v, σ, σ_h is h. These points are therefore given by

$$\rho_h=x_u+\left(h\frac{\partial}{\partial u}\log\frac{S}{M}\right)x,\quad \sigma_h=x_v+\left(h\frac{\partial}{\partial v}\log\frac{S}{N}\right)x.$$

The points thus defined are called the differential generating points, and the line determined by them the differential generating line. The congruence of these lines is similarly named. The following geometric characterization of isothermally conjugate nets is discovered in terms of a property described by the reviewer [Trans. Amer. Math. Soc. **46**, 389–409, p. 396 (1939); these Rev. **1**, 85] of a congruence being central to the surface S. The differential generating congruence (of lines $\rho_h\sigma_h$) is central to S if and only if the conjugate net $M^2du^2-N^2dv^2=0$ is isothermally conjugate. The notion is extended to a characterization of isothermally orthogonal nets by referring the surface to an orthogonal net as parametric and replacing the R-harmonic line in the above stated characterization by the ideal line in the tangent plane to S at x. The line $\rho\sigma$ obtained in this way is found to generate a congruence which is central to S with respect to the orthogonal parametric net if and only if $(\partial^2/\partial v\partial u)(\log M/N)=0$. If $M=E^{\frac{1}{2}}$, $N=G^{\frac{1}{2}}$, this becomes the condition that the parametric net be isothermally orthogonal (isometric). The net $E\,du^2-G\,dv^2=0$ is, of course, that orthogonal net whose tangents at x separate harmonically the tangents to the given orthogonal parametric net. As a final application the following geometric characterization of the classical canonical lines is derived. If the integral generating line is the R_λ-associate of the reciprocal of the projective normal, then the curve C_λ is a curve of Segre and the differential generating line is any one whatever of the classical canonical lines of the second kind. [For the definition of the R_λ-associate of a line see the reviewer's paper cited above, p. 390.]

P. O. Bell (Lawrence, Kans.).

Lalan, V. Sur l'emploi d'un repère canonique dans l'étude des réseaux conjugués. Bull. Soc. Math. France **78**, 162–184 (1950).

This is the detailed account of results announced earlier [C. R. Acad. Sci. Paris **229**, 1115–1116 (1949); these Rev. **11**, 395]. Given a conjugate net on a surface, the author attaches a frame $Me_1e_2e_3$ with e_1, e_2 tangent to the curves of the net and e_3 a unit normal to the surface. The frame is called canonical with respect to the conjugate net if relative to it the asymptotic form is a sum of two squares. Using the method of É. Cartan, the author establishes the formal apparatus for the theory of conjugate nets relative to canonical frames. With a ds^2 and a conjugate net given, the determination of the canonical frames is in general unique. There are exceptional cases where there are two or an infinite number of solutions. In the latter case the conjugate net is called persisting. Surfaces carrying such a net depend on six arbitrary functions in one variable. The determination of such surfaces depends on the solution of a partial differential equation of the fourth order, which the author derives explicitly.

S. Chern (Chicago, Ill.).

Ōtsuki, Tominosuke. On some 4-dimensional Riemannian spaces. I. Mem. Fac. Sci. Kyūsyū Univ. A. 4, 193–212 (1949).

The author studies four-dimensional Riemannian spaces which admit certain holonomy groups and gives the canonical forms of their line elements. *S. Chern.*

Ōtsuki, Tominosuke. On some 4-dimensional Riemannian spaces. II. Mem. Fac. Sci. Kyūsyū Univ. A. 4, 213–217 (1949).

A continuation of the paper reviewed above.

S. Chern (Chicago, Ill.).

Verbickiĭ, L. L. On the equations for imbedding Riemannian spaces of class 2 in Euclidean spaces. Trudy Sem. Vektor. Tenzor. Analizu. 8, 425–429 (1950). (Russian)

The Gauss-Codazzi-Ricci relations for a Riemannian V_n in a Euclidean R_{n+k} contain, apart from the Riemann-Christoffel tensor R_{ijlm}, the symmetrical tensors $\overset{(\sigma)}{b}_{ij}$ ($\sigma=1, 2, \cdots, k$; $i, j=1, 2, \cdots, n$) and the $\frac{1}{2}k(k-1)$ vectors $\overset{ab}{T}_i=-\overset{ba}{T}_i$ ($a, b=1, 2, \cdots, k$). The smallest number k for given ds^2 is called the class. This paper proves two theorems for V_n of class $k=2$. (i) If the tensors $\overset{(a)}{b}_{ij}$ and $\overset{ab}{T}_i$ ($a, b=1, 2$) satisfy the Gauss-Codazzi equations, and if at least one of the two tensors $\overset{(a)}{b}_{ij}$ has rank $r>2$, then these tensors satisfy also the equations of Ricci. (ii) If the rank r of one of the tensors $\overset{(a)}{b}_{ij}$ is greater than 3, then the Codazzi equation for this tensor is satisfied in consequence of the Gauss equation and the Codazzi equation for the other tensor $\overset{(a)}{b}_{ij}$. [It might be useful to compare these results with those of A. Schwarz, J. Math. Physics 20, 30–79 (1941); these Rev. 2, 303.] *D. J. Struik* (Cambridge, Mass.).

***Norden, A. P. Prostranstva affinnoĭ svyaznosti. [Affinely Connected Spaces].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 463 pp.

The author's avowed aim was to write a text-book for "aspiring geometers because the existing treatises are not suitable to that purpose." In spite of the great wealth of material contained in its pages the reviewer does not believe this purpose is achieved. The guiding principle of the book is Klein's Erlanger program except that instead of the projective space and its subspaces the author uses vector spaces. The geometries considered besides projective are affine, equi-affine, centro-affine, non-Euclidean, Euclidean, pseudo-Euclidean, and conformal. Parallelism is defined in terms of a segment of two points and an affine connection is defined in terms of a parametrisation of the space. A whole chapter is devoted to various special connections and to pairs of related connections. Most of the geometrical concepts are defined analytically; thus lines of curvature are lines given by certain differential equations. A large part of the book is devoted to normalized spaces, a subject to which the author has made many significant contributions. The last three chapters are an excellent account of the geometry of various spaces of two and three dimensions and the properties of various nets and congruences. There are also a number of applications to classical physics which seem to the reviewer to be inessential.

M. S. Knebelman (Pullman, Wash.).

Širokov, P. A. Projectively Euclidean symmetric spaces. Trudy Sem. Vektor. Tenzor. Analizu. 8, 73–81 (1950). (Russian)

A connection is projectively Euclidean if

$$\Gamma^{\gamma}_{\alpha\beta}=\delta_\beta{}^\gamma\psi_\alpha+\delta_\alpha{}^\gamma\psi_\beta;$$

it is called symmetric if the curvature tensor $B^\alpha{}_{\beta\gamma\delta}$ satisfies the equation $B^\alpha{}_{\beta\gamma\delta,\,\varepsilon}=0$. This is equivalent with $\omega_{\alpha\beta,\,\gamma}=0$, $\omega_{\alpha\beta}=\partial_\beta\psi_\alpha-\psi_\alpha\psi_\beta$. It is now shown that ψ_α must be a gradient vector $\psi_\alpha=\partial_\alpha\psi$, where $\psi=-\frac{1}{2}\log\varphi$, and φ is an arbitrary quadratic function $\varphi=a_{\alpha\beta}x^\alpha x^\beta+a_\alpha x^\alpha+a$. Since $B^\alpha{}_{\beta\gamma\alpha}=(n-1)\omega_{\beta\gamma}$ the connection is equiaffine. The tensor $\omega_{\alpha\beta}$ is determined but for a constant multiplier. If and only if

$$\Delta=\begin{vmatrix} a & a_1 & \cdots & a_n \\ a_1 & a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n1} & \cdots & a_{nn} \end{vmatrix}\neq 0,$$

the space is Riemannian (and of constant curvature).

If collineations are transformations which transform geodesics into geodesics, then, by the use of a result of Eisenhart [Non-Riemannian geometry, Amer. Math. Soc. Colloquium Publ. no. 8, New York, 1927, pp. 126, 127] it is found that they are those projective transformations (characterized by $\delta x^\alpha=v^\alpha\delta t$, $v^\alpha=p^\alpha+q_\sigma{}^\alpha x^\sigma+x^\alpha r_\sigma x^\sigma$, p, q, r constants) for which the "absolute" $\varphi=\sum a_{ij}x^i x^j+\gamma=0$ ($i, j=1, \cdots, m$, $m\leqq n$, $a_{ij}=\alpha_i\delta_{ij}$, $\alpha_i{}^2=1$, γ constant $\neq 0$) remains invariant. The paper ends with a remark by A. P. Norden that in these spaces with a quadratic form φ a polar correlation is determined with respect to the hypersurface $\varphi=0$ [cf. A. P. Norden, Mat. Sbornik N.S. 20(62), 263–281 (1947); these Rev. 9, 67]. *D. J. Struik* (Cambridge, Mass.).

Bol, G. Zur tensoriellen Behandlung der projektiven Flächentheorie. Math. Ann. 122, 279–295 (1950).

If one deals with the projective differential geometry of an $(n-1)$-dimensional hypersurface in an n-dimensional space, there are two main problems to be considered: the factor of proportionality involved in the definition of homogeneous coordinates; and the projective normal. The first problem leads in a natural way either to the Weyl connection or to a projective connection and these connections lead to the projective normal. [Cf. the reviewer's papers, Rend. Circ. Palermo 57, 402–430 (1933), where the large space is supposed to be a linear one, and Ann. of Math. (2) 39, 725–761 (1938), where the large space is a projective curved one.] In the present paper the author starts with an arbitrarily given projective normal (and its dual counterpart) and an arbitrarily given affine connection. Dealing with the factor of proportionality he uses the pseudotensors and their covariant derivatives as introduced by Schouten and the reviewer [Math. Z. 30, 414–432 (1929)] and obtains a set of fundamental (Gauss) equations and their conditions of integrability. Since these formulae are valid for an arbitrary connection they also hold for the Weyl connection mentioned above; this leads in a natural way to the projective normal and to Darboux hyperquadrics [cf. also the reviewer, Časopis Pěst. Mat. Fys. 66, 229–260 (1937)]. The last two sections deal with the application of these results in the case $n=3$. *V. Hlavatý* (Bloomington, Ind.).

Vagner, V. V. Classification of linear connections in a composite manifold $X_{n+(1)}$ according to their holonomy group. Trudy Sem. Vektor. Tenzor. Analizu. 7, 205–226 (1949). (Russian)

A compound manifold $X_{n+(m)}$ consists of a basic n-dimensional space X_n with coordinates ξ^α, with each point of which

is associated an m-dimensional manifold $X_{(m)}$ with coordinates η^α. The transformations and mappings to which $X_{n+(m)}$ is subjected are defined by $\bar{\xi}^\alpha = F^\alpha(\xi)$, $\bar{\eta}^\alpha = \varphi^\alpha(\xi, \eta)$, the transformations being nonsingular. The space $X_{n+(m)}$ is linearly connected if $d\eta^\alpha$ are linear in $d\xi^\alpha$, i.e., $d\eta^\alpha + \Gamma_\alpha{}^\alpha(\xi, \eta)\, d\xi^\alpha = 0$. From the components of linear connection one obtains the curvature vector $R^\alpha = 2([\partial\Gamma^\alpha] - [\Gamma^\alpha \partial_\alpha \Gamma^\alpha])$, where $\partial = d\xi^\alpha \partial/\partial\xi^\alpha$ and $\partial_\alpha = \partial/\partial\eta^\alpha$. The components R^α are skew symmetric bilinear forms. The holonomy group of the space is defined in the usual manner and, as is usual in differential geometry, the transformations corresponding to infinitesimal coordinate contours are considered. The group generated by the vector R^α depends on the class of this vector: the class being k, $R^\alpha = R_i{}^\alpha \Omega^i$, $i = 1, \cdots, k$, where Ω^i are independent skew symmetric bilinear forms in X_n with coefficients independent of η, and $R_i{}^\alpha$ are contravariant vectors in $X_{(m)}$, linearly independent with coefficients depending on ξ at most. The classification of $X_{n+(1)}$ according to the holonomy group can be obtained because all geometric objects in $X_{(1)}$ are known. They are (1) densities of various weights, (2) affine connections ($\bar{\gamma} = \varphi'^{-1}\gamma - \varphi''\varphi'^{-2}$), and (3) projective connections ($\bar{p} = \varphi'^{-2}p - \varphi'''\varphi' + \frac{3}{2}\varphi''^2\varphi'^{-4}$). If the holonomy group is the identity, the space is without curvature and there exists a coordinate system in which $\Gamma = 0$. If the group is of one parameter, coordinates can be chosen so that $\Gamma = \Gamma_\alpha(\xi)\, d\xi^\alpha$. If the group is of two parameters,

$$\Gamma = \underset{1}{\eta}\Gamma_\alpha(\xi)\, d\xi^\alpha + \underset{0}{\Gamma}_\alpha(\xi)\, d\xi^\alpha.$$

For a three-parameter group, corresponding to a projective connection, Γ is quadratic in η with coefficients which are linear differential forms in X_n. Finally the group may be infinite. The last part of the paper solves the equivalence problem for $X_{2+(1)}$ from the above point of view.

M. S. Knebelman (Pullman, Wash.).

Yano, Kentarô, and Imai, Tyiûti. On affine collineations in projectively related spaces. J. Math. Soc. Japan **1**, 287–288 (1950).

The authors give simplified proofs of two theorems previously proved by the reviewer, making use of Lie derivatives. Their notation is not in conformity with the accepted one; X_α as used in the note is the partial Lie derivative and not an infinitesimal generator of the group (they are identical only for scalar functions). *M. S. Knebelman.*

Ržehina, N. F. On the theory of fields of local curves in X_n. Doklady Akad. Nauk SSSR (N.S.) **72**, 461–464 (1950). (Russian)

A discussion of the invariants of a curve $x^\alpha = l^\alpha(\eta)$, $\alpha = 1, \cdots, n$, in a centro-affine space E_n. Assuming that the curve does not lie in a hyperplane through the center of E_n, the vector l^α and its first $n-1$ derivatives are independent so that $d^n l^\alpha/d\eta^n + \sum_{i=1}^{n} \Omega^i d^{n-i} l^\alpha/d\eta^{n-i} = 0$. The Ω's are the components of a geometrical object of class n in the space X_1 associated with the curve. If $\gamma = [2/n(n-1)]\Omega^1$, then γ can be taken as an affine connection in X_1 by means of which the Ω's may be replaced by covariant derivatives of densities in X_1 giving $\nabla^n l^\alpha + \overset{(2)}{\omega}\nabla^{n-2} l^\alpha + \cdots + \overset{(n-1)}{\omega}\nabla l^\alpha + \overset{(n)}{\omega} l^\alpha = 0$. If $\overset{(n)}{\omega} \neq 0$, the affine arc and the successive affine curvatures of the curve are expressible in terms of the ω's. If $\overset{(n)}{\omega} = 0$, the curve lies in a hyperplane so that the above method may be used for the study of curves in general affine space. The author indicates the solution of the equivalence problem in terms of the above densities $\overset{(k)}{\omega}$ and points out that this theory may be applied to the variational problem of Lagrange with $n-2$ conditions. *M. S. Knebelman* (Pullman, Wash.).

Vedernikov, V. I. The conformal applicability of surfaces. Doklady Akad. Nauk SSSR (N.S.) **73**, 437–440 (1950). (Russian)

The conformal geometry of a surface in a three-dimensional Möbius space is determined by means of its normalization. Such normalization determines a geometry of Weyl on the surface, its inner geometry, and the conformal applicability of two normalized surfaces is expressed in terms of this geometry. The author gives three criteria for conformal applicability. One of them is the familiar one, viz., two normalized surfaces are conformally applicable if they have the same Riemann angular metric on their central spheres.

M. S. Knebelman (Pullman, Wash.).

***Sasaki, Shigeo. Kyôkei setsuzoku kikagaku. [Geometry of Conformal Connection].** Kawade-shobô, Tokyo, 1948. 3+3+265 pp. 250 Yen.

H. Weyl [Math. Z. **2**, 384–411 (1918); Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. **1921**, 99–112] opened the way to the conformal differential geometry of Riemannian spaces in which one studies the properties of the spaces invariant under the so-called conformal transformation $\bar{g}_{jk} = \rho^2 g_{jk}$ of the Riemannian metric. He discovered a tensor, now called Weyl's conformal curvature tensor, whose vanishing is a necessary condition that the space be conformally flat, that is to say, that the space can be mapped conformally on the Euclidean space. That this is also sufficient was proved by Schouten [Math. Z. **11**, 58–88 (1921)].

Since the relative tensor G_{jk} defined by $g_{jk}/g^{1/n}$, where g is the determinant $|g_{jk}|$, is invariant under a conformal transformation of g_{jk}, the conformal geometry of a Riemannian space may be reduced to the study of the differential invariants of the relative tensor G_{jk}. This point of view was adopted by the geometers of the American school: J. Levine, J. M. Thomas, T. Y. Thomas, J. L. Vanderslice, and O. Veblen [cf., e.g., T. Y. Thomas, The Differential Invariants of Generalized Spaces, Cambridge University Press, 1934]. On the other hand, É. Cartan [Ann. Soc. Polon. Math. **2**, 171–221 (1924)] has introduced the idea of a space with conformal connection. In spite of its beauty and fruitfulness, the paper of Cartan seems to have remained rather isolated. The reviewer [C. R. Acad. Sci. Paris **206**, 560–563 (1938); J. Fac. Sci. Imp. Univ. Tokyo Sect. 1. **4**, 1–59 (1939); these Rev. **1**, 88] showed that, if the conformal connection is normal (in Cartan's sense) and the frame of reference is natural (also in Cartan's sense), then the theory of spaces with conformal connection will coincide with that studied by J. M. Thomas, T. Y. Thomas, and others; and if the connection is normal and the frame of reference is the so-called Veblen's repère, then the theory of spaces with conformal connection will coincide exactly with the conformal geometry of Riemannian spaces. However, all the writers mentioned above studied exclusively the conformal properties of a Riemannian space itself and paid only slight attention to the conformal properties of curves and surfaces immersed in a Riemannian space. S. Sasaki, Y. Muto, and the reviewer have developed, since 1938, the conformal theory of curves and surfaces in a conformally connected space as well as in a Riemannian

space. S. Sasaki has obtained also a result on the structure of a conformally connected space whose group of holonomy fixes a point or a hypersphere. Conformal geometry was also studied, using various other methods, by H. W. Brinkmann, A. Fialkow, St. Gołab, J. Haantjes, W. Wrona, A. Haimovici, V. Hlavatý, V. A. Hoyle, M. S. Knebelman, A. D. Michal, J. A. Schouten, A. Taub, and others. This book contains almost all the results mentioned above in the geometry of conformal connection. The author completed the manuscript of this book in 1943, but the book was not published until 1948.

In chapter 1 entitled "Preliminaries to generalization of the idea of space," the author explains the Möbius transformation (conformal transformation), treating it projectively. Next, he explains so-called Klein geometries in order that the readers may understand what position conformal geometry occupies in the classification of geometries in Klein's Erlanger Programm. Finally, he treats the Möbius' conformal geometry by the method of the moving frame of reference of É. Cartan. Here appear the famous equations of structure of the Möbius space. In chapter 2, entitled "Spaces with conformal connection," the author first gives an intuitive definition of these spaces and of the development of tangent Möbius spaces along a curve. To develop fully the theory of spaces with conformal connection, one must have sufficient knowledge of Riemannian geometry and tensor calculus. Consequently the author explains here the elements of Riemannian geometry by the method of the moving frame of reference. He next outlines the tensor calculus. He defines the Riemannian space as a space with Euclidean connection (without torsion), and to illustrate the idea of connection, gives a detailed exposition of Levi-Civita's parallelism. Some intuitive examples are given. The author then turns to spaces with conformal connection. He shows that, among the so-called semi-natural frames of reference, we can choose two most convenient ones, Veblen's repère and the repère naturel. He next explains the conformal tensors, conformal covariant derivatives, and conformal curvature tensors. He then discusses the spaces which satisfy the equations of structure and also the conformal geometry of Riemannian spaces. He next defines the spaces with normal conformal connection and shows that the conformal geometry of Riemannian spaces and the theory of spaces with normal conformal connection are essentially the same. Finally, he discusses the relation between the conformal geometry of conformally flat Riemannian spaces and Möbius' conformal geometry. He emphasizes that these are not the same geometries.

In chapter 3, entitled "Equivalence problems," he defines the semi-equivalence (meriedrical isomorphism) and the equivalence (holoedrical isomorphism) between two spaces with conformal connection, and gives the solutions of these equivalence problems in terms of the complete conformal derivatives of J. L. Vanderslice [T. Y. Thomas also solved this problem]. Chapter 4 is devoted to the theory of curves in a space with conformal connection. He establishes the Frenet formulas for a curve in a space with conformal connection, first by the use of a projective parameter and next by the use of a purely conformal parameter. Then, in the light of these formulas, he discusses the conformal circles and generalized loxodromes found by Y. Muto and the reviewer. Chapter 5 is devoted to the theory of hypersurfaces. To choose the most convenient frame of reference for the study of hypersurfaces, he defines first the so-called central hypersphere and proves its conformal character. He then discusses successively the induced conformal connection, equations of Gauss and Codazzi, umbilical hypersurfaces, the condition that the induced connection be normal, theorems on conformal circles and umbilical hypersurfaces, self-concurrent curves, and finally the spaces with normal conformal connection admitting a family of umbilical hypersurfaces.

In the last chapter (6), the author develops the theory of groups of holonomy of a space with normal conformal connection. After having stated the definition of group of holonomy and a fundamental theorem of É. Cartan, he proves his own theorem: If the group of holonomy of a space with normal conformal connection is a subgroup of the general Möbius group and fixes a point or a hypersphere, then the space is a space with normal conformal connection which corresponds to a family of Riemannian spaces conformal to an Einstein space with vanishing or nonvanishing scalar curvature. He gives an application of this theorem to the results of H. W. Brinkmann on Riemannian spaces conformal to an Einstein space. He further discusses the case in which the group of holonomy fixes two points or hyperspheres and its relation to the results of H. W. Brinkmann on Einstein spaces conformal to each other. He finally proves the existence of an umbilical hypersurface in a complete space with normal connection whose group of holonomy fixes a real hypersphere, and a generalization of the representation of Poincaré to an Einstein space. As the book was written in the early 1940's, it does not contain the recent work on conformal geometry by J. Haantjes and A. Fialkow. The exposition is clear and intuitive. Each chapter contains many suggestive questions solved or unsolved. *K. Yano* (Princeton, N. J.).

Laptev, G. F. On manifolds of geometric elements with a differential connection. Doklady Akad. Nauk SSSR (N.S.) **73**, 17–20 (1950). (Russian)

Let the fundamental group of an n-dimensional space E be G_r defined by $dx^i=\xi_s^i(\alpha)\cdot\vartheta^s(v, dv)$, $i, j, k=1, \cdots, n$; $p, q, s=1, \cdots, r$. In E there is a geometrical object F whose stationary subgroup g, by a suitable choice of the basic invariants ϑ, may be determined by the completely integrable Pfaffian system $\vartheta^{s_1}=0$, $p_1, s_1=1, \cdots, r_1\leqq r$. In that case the structure constants $c_{p_2q_2}^{p_1}=0$, $p_2, q_2=r_1+1, \cdots, r$. The space E with the object of support F is now regarded as an element and the author considers them as elements of a topological N-dimensional cube, i.e., $EF(u)$ are functions of N parameters $u^1, \cdots, u^N$. The mapping of one element upon a neighboring one is then determined by

$$x^i(u+du)\to x^i(u, du)=x^i(u)+\xi_s^i(x)\omega^s(u, du)+\rho e,$$

where $\omega^s=\Gamma_j^s(u)\,du^j$. Since v^s are the parameters of the fundamental group, x^i are functions of u and v so that the complete mapping is given by

$$x^i(u+du; v+dv)\to x^i(u, v)+\xi_{s_{12}}^i(x(u, v))\omega^{s_{12}}(u, v, du, dv)+\overline{\rho e},$$

where $\omega^{s_{12}}=\Gamma_{p_{12}}^{s_{12}}dv^{p_{12}}+\Gamma_j^{s_{12}}du^j$ (why the author introduces p_{12}, q_{12}, s_{12} for p, q, s is not apparent). The differential forms $\omega^{s_{12}}$ must satisfy certain integrability conditions which lead to the following six tensors: a torsion tensor; a principal torsion tensor; a collateral torsion tensor; and three similar tensors for curvature. These tensors satisfy generalized Bianchi identities. *M. S. Knebelman* (Pullman, Wash.).

Ghosh, N. N. On complex tensor calculus. Bull. Calcutta Math. Soc. **42**, 65–72 (1950).

Given is a space with coordinates

$$x^\mu = q^\mu + ip^\mu,$$

$\mu = 1, 2, \cdots, n$, $i = \sqrt{-1}$, with functions $M(x^\mu)$ which are single valued, continuous in q, p, and sufficiently differentiable. If $\xi_\mu = \frac{1}{2}(\partial/\partial q^\mu + i\partial/\partial p^\mu)$ and $\xi_{\mu^*} = \frac{1}{2}(\partial/\partial q^\mu - i\partial/\partial p^\mu)$, then $dM = dx^\mu(\xi_\mu M) + dx^{\mu^*}(\xi_{\mu^*} M)$, which is written in the form $dM = dx^\mu(\xi_\mu M) = dx^{\mu^*}(\xi_{\mu^*} M)$ by summing on $2n$ terms. Then $\xi_\mu x^\nu = 1$, $\xi_{\mu^*} x^{\nu^*} = 1$ if $\mu = a$, $\xi_{\mu^*} x^{\nu^*} = 0$ if $\mu \neq a$. A tensor calculus is developed under the transformations

$$q'^\mu = q'^\mu(q^1, q^2, \cdots, q^n, p^1, p^2, \cdots, p^n),$$
$$p'^\mu = p'^\mu(q^1, q^2, \cdots, q^n, p^1, p^2, \cdots, p^n).$$

Then tensors are defined by equations such as

$$A'^{\mu\nu} = A^{\alpha\beta}(\xi_\alpha x'^\mu)(\xi_\beta x'^\nu), \quad A'_{\mu\nu} = A_{\alpha\beta}(\xi_\mu' x^\alpha)(\xi_\nu' x^\beta).$$

A metric is established by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^{\nu^*} \qquad (g_{\mu\nu} = g_{\nu\mu},\ g_{\mu^*\nu} = g_{\nu\mu^*} = \text{conj. } g_{\nu\mu^*}).$$

With the aid of the determinant of order $2n$

$$g = \begin{vmatrix} g_{1\cdot 1} & \cdots & g_{1\cdot n} & g_{1\cdot 1^*} & \cdots & g_{1\cdot n^*} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ g_{n1} & \cdots & g_{nn} & g_{n1^*} & \cdots & g_{nn^*} \end{vmatrix}$$

tensors $g^{\mu\nu}$ and $g^{\mu\nu^*}$ can be defined such that $g_{\mu\cdot}g^{\nu^*\cdot} = g^\nu{}_\mu = 1$ $(\mu = \nu)$ and $= 0$ $(\mu \neq \nu)$, $g_{\mu\cdot}g^{\nu^*\cdot} = 0$. Christoffel symbols and covariant derivatives are defined as follows:

$$\{{}^\sigma_{\mu\nu}\} = g^{\lambda^*\sigma}[\mu\nu, \lambda] = \tfrac{1}{2}g^{\lambda^*\sigma}(\xi_\nu g_{\mu\lambda} + \xi_\mu g_{\nu\lambda} - \xi_\lambda g_{\mu\nu}),$$
$$A^\mu{}_{,\nu} = \xi_\nu A^\mu + \{{}^\mu_{\alpha\nu}\}A^\alpha,$$

and a curvature tensor by $A_{\mu,\nu\rho} - A_{\mu,\rho\nu} = A_{\sigma\cdot}R^\sigma{}_{\mu\nu\rho}$. An application is made to relativity by taking $g_{\mu\nu} = 0$, $\Re g_{\mu\nu^*} = 0$ $(\mu, \nu = 1, 2, 3, 4)$. The metric determined by $g_{\mu\nu^*}$ (the electromagnetic metric tensor) $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu + g_{\mu\nu^*}\, dx^\mu\, dx^{\nu^*}$ is preserved under a Lorentz transformation. The Maxwell equations are expressed by $g_{\mu\nu^*} = i(\xi_\nu A_\mu - \xi_\mu A_\nu)$, $(\xi_\nu A_\mu - \xi_\mu A_\nu)$ real, $\xi_\mu h^{\mu\nu^*} = J^\nu$, where $h^{\mu\nu^*} = A^{\mu\nu}A^{\lambda^*\nu^*}g_{\lambda\nu^*}$ $(A^{11} = A^{22} = A^{33} = 1$, $A^{44} = -1$, $A^{\mu\nu} = 0$, $A^{\mu^*\nu^*} = 0)$. Here A_μ is the electromagnetic potential and J_μ is the charge-current vector.

D. J. Struik (Cambridge, Mass.).

NUMERICAL AND GRAPHICAL METHODS

Grammel, R. Tafeln der verallgemeinerten Kreisfunktionen Sin $(4)v$, Cos $(4)v$, Sin $(6)v$, Cos $(6)v$, Sin $(8)v$, Cos $(8)v$. Ing.-Arch. **18**, 250–254 (1950).

The functions $x = \cos(n)v$ and $y = \sin(n)v$ were defined in an earlier paper [same Arch. **16**, 188–200 (1948); these Rev. **10**, 38] as the inversion of the integrals

$$v = \int_x^1 f_n(t)\, dt = \int_0^y f_n(t)\, dt,$$

$f_n(t) = (1 - t^n)^{(1-n)/n}$, and have periodic properties with half-period $\pi_n = 2\int_0^1 f_n(t)\, dt$. In the present paper 6D tables of x and y are given for $n = 4, 6, 8$, $v = 0.00(0.01)1.00(0.1)4.0$ and for $2v/\pi_n = 0.00(0.05)1.00$. *A. Erdélyi.*

Uhler, Horace S. Many-figure approximations to $\sqrt{2}$, and distribution of digits in $\sqrt{2}$ and $1/\sqrt{2}$. Proc. Nat. Acad. Sci. U. S. A. **37**, 63–67 (1951).

van Wijngaarden, A. Rounding-off errors. Math. Centrum, Amsterdam. Rapport **ZW-1950-001**, 13 pp. (1950). (Dutch)

This is a preliminary report of researches of the author and W. L. Sheen. Consider rounding to integers: write $f = F + \varphi$ where $F \equiv 0 \pmod 1$ and $|\varphi| < \frac{1}{2}$ or where $F \equiv 0 \pmod 2$ and $|\varphi| = \frac{1}{2}$; also write $F = Af$. Let $a_1, \cdots, a_n$ be given constants. Let the quantities $f_1, f_2, \cdots$ be such that the corresponding $\varphi_1, \varphi_2, \cdots$ are homogeneously distributed and such that any n successive φ_i are independent. The following two questions are discussed: (1) What is the distribution of the actual error $\psi = \sum a_k f_k - \sum a_k F_k$ and (2) what is the distribution of $P = A\sum a_k f_k - A\sum a_k F_k$? The case when the a_k are such that $\sum a_k f_k$ is formally the nth difference of the f_k is discussed in detail. [See also J. C. P. Miller, Math. Tables and Other Aids to Computation **4**, 3–11 (1950); these Rev. **12**, 361]. *J. Todd.*

Tienstra, J. M. The foundation of the calculus of observations and the method of least squares. Bull. Géodésique N.S. **1948**, 289–306 (1948).

Expository article in which the exposition leaves much to be desired. *B. Epstein* (Detroit, Mich.).

Cohen, E. Richard. Nonindependent observational equations in the theory of least squares. Physical Rev. (2) **81**, 162 (1951).

Hartree, D. R. Automatic calculating machines. Math. Gaz. **34**, 241–252 (4 plates) (1950).

Expository lecture given before the Mathematical Association in January, 1950.

Eberl, Walther. Zur Multiplikation reeller Zahlen. Z. Angew. Math. Physik **1**, 411–413 (1950).

A method is given for shortening the multiplication of two real numbers, particularly with a calculating machine, and a bound is stated for the difference between the exact and the shortened product. *E. Frank* (Chicago, Ill.).

Borel, Émile. Le calcul numérique des séries divergentes. C. R. Acad. Sci. Paris **232**, 457–458 (1951).

The author suggests the desirability of a study of the following question. Let $\sum u_n$ be a convergent series, such as the power series expansion of $\cos x$ when $x = 20$, for which the partial sums of low order are very different from the number L to which the series converges. How can one use the first few terms of the series to obtain good approximations to L with estimates of errors? *R. P. Agnew.*

Stosick, A. J. A method for indexing powder photographs, using linear Diophantine equations, and some tests for crystal classes. Acta Cryst. **2**, 271–277 (1949).

R. Hesse [Acta Cryst. **1**, 200–207 (1948)] has developed a numerical, as contrasted with graphical, method for the indexing of powder photographs of tetragonal and hexagonal classes of crystals. The method stems from expressing $q = \sin^2\theta$ (the sin θ of Bragg's law) as a 2-dimensional vector over the field of rational numbers, whereas q in the standard theory is regarded as a quadratic form. In the present paper the vector q is shown to have dimension 1, 2, 3, 4, 6 in the cubic, tetragonal, or hexagonal, orthorhombic, monoclinic, triclinic classes of crystals, respectively. It then follows that a test for crystal class can be obtained from the linear dependence of sets of q values. The dimension, and hence the class, is decided by finding the rank of the matrix of integral coefficients of linear relations among various experi-

mentally determined q values. The Hesse method is also extended by the use of linear Diophantine analysis, which gives an alternative numerical procedure. Finally, by applying such quadratic Diophantine analysis as the theory of the simultaneous equations $x^2+y^2=z^2$, $y^2+z^2=t^2$ to the quadratic form q, a criterion is developed for assigning a crystal to one of three types: cubic; tetragonal, hexagonal; orthorhombic, monoclinic, triclinic. *I. Niven.*

Stefaniak, H. St. Ein graphisches Verfahren zur Bestimmung der Zeitkonstanten und der Schwingungsdauer eines linearen Systems dritter Ordnung. Ing.-Arch. 18, 222–232 (1950).

Nomograms for the determination of the roots of a cubic equation that occurs as the characteristic equation for certain linear control systems. *M. Golomb.*

Mayot, Marcel. Recherche des valeurs et directions propres d'une matrice. Ann. Astrophysique 13, 282–293 (1950).

By an elementary change of variables, the system of equations $\sum_j(a_{ij}-s\delta_{ij})x_j=0$, $i=1, \cdots, n$, can be transformed into a set, $-sX_i+X_{i+1}=0$, $i=1, \cdots, n-1$, $p_nX_1+p_{n-1}X_2+\cdots+(p_1-s)X_n=0$, for which the determinant is essentially $(s^n-p_1s^{n-1}-p_2s^{n-2}-\cdots-p_n)$. Thus this "method of substitution" yields a convenient method for the calculation of the coefficients in the secular equation. A matrix procedure for this change of variables is given and a comparison with other numerical methods for this purpose is given. *F. J. Murray* (New York, N. Y.).

Ostrowski, Alexandre. Un théorème d'existence pour les systèmes d'équations. C. R. Acad. Sci. Paris 231, 1114–1116 (1950).

The author gives sufficient conditions for the existence of a solution of a system of equations $y_\nu=f_\nu(x_1, \cdots, x_n)$ in a neighborhood of a point where the equations are approximately satisfied. This result is applicable in the case where numerical procedures which stop at a given approximation are used. The conditions involve $\partial x_j/\partial y_\nu$.
F. J. Murray (New York, N. Y.).

Birkhoff, Garrett, and Young, David. Numerical quadrature of analytic and harmonic functions. J. Math. Physics 29, 217–221 (1950).

Les auteurs proposent la formule:

$$\int_{z_0}^{z_4} F(z)\,dz=(2h/15)[24F_0+4(F_1+F_3)-F_2-F_4],$$

F_0, F_1, F_2, F_3, F_4 valeurs de la fonction analytique $F(z)$ aux points d'affixe z_0, z_1, z_2, z_3, z_4 (z_0 est le centre d'un carré de demi diagonale h dont z_1, z_2, z_3, z_4 sont les sommets). L'erreur est majorée par $(h^7/1890)\times\{\text{borne supérieure de } F^{(6)} \text{ dans le carré}\}$. Les auteurs comparent cette formula à celle de Simpson. La formule s'applique également à l'intégration des fonctions harmoniques. *J. Kuntzmann.*

Haasbroek, N. D. A nomogram for

$$F_0^z=\frac{1}{2}\cdot\frac{2}{\sqrt{\pi}}\int_{t=0}^{t=\frac{z}{m\sqrt{2}}}e^{-t^2}\,dt.$$

Bull. Géodésique N.S. 1948, 285–288 (1 plate) (1948).

If the scale of x is placed so that $X=x$ on the X-axis and the scale of $f^{-1}(y)$ is placed on the Y-axis of a Cartesian system of coordinates (X, Y) in the plane, the value of $f(x/\sigma)$ can be read by projecting x first on the line $\sigma Y-X=0$ and then on the Y-axis. The present paper gives the details for constructing this nomogram when f is the probability integral and σ the standard deviation. A projective transformation is used to improve the spacing and a drawing is given for the range $0.3\leqq\sigma\leqq3$, $0\leqq x\leqq\sigma$. With a slide rule to calculate x/σ, an operator will find the standard tables just as easy to use, more accurate, and applicable to a more extended range. *J. M. Thomas* (Durham, N. C.).

Hartree, D. R. A method for the numerical integration of first-order differential equations. Proc. Cambridge Philos. Soc. 46, 523–524 (1950).

The author suggests a new formula for the numerical step-by-step integration of the first order differential equation $y'=f(x, y)$. If the solution y is tabulated at equidistant intervals δx up to an argument $x=0$ (say) with $y(0)=y_0$, the next value $y_1=y(\delta x)$ is obtained by computing in turn $y''(0)=f_x(0, y_0)+f_y(0, y_0)f(0, y_0)=y_0''$, $\delta^2y_0=(\delta x)^2y_0''$, and hence y_1, $y'(\delta x)=y_1'$, $y''(\delta x)=y_1''$;

$$\delta y_1=\tfrac{1}{2}(\delta x)\{y_0'+y_1'-\tfrac{1}{6}\delta x(y_1''-y_0'')\}.$$

The last equation then provides from $y_1=y_0+\delta y_1$ an improved value of y_1 to within an accuracy of order $O(\delta x^4)$. The use of additional estimated higher order terms in all formulae is discussed and the accuracy of the procedure is found to be superior to that of methods normally employed. A similar treatment of the second order equation $z''=f(x, z, z')$ is indicated. *H. O. Hartley* (London).

Todd, John. Notes on modern numerical analysis. I. Solution of differential equations by recurrence relations. Math. Tables and Other Aids to Computation 4, 39–44 (1950).

The solution of the differential equation (1) $y''=-y$ by finite differences is analyzed in detail in order to determine the cause of spurious results whenever certain types of difference equivalents are used. In particular, it is found that the substitution (2) $y''=h^{-2}\delta^2y$ and (3) $y''=h^{-2}(1+h^2/12)\delta^2y$ give generally valid results; while the use of the more accurate difference representation (4) $y''=h^{-2}(\delta^2-\delta^4/12)y$ yields numerical results which tend to infinity. An explanation of this rather unexpected result is contained in an analysis of the nature of the characteristic roots of the finite difference equations obtained by the substitution of equations (2), (3), and (4) in equation (1). *H. Polachek* (White Oak, Md.).

Hulthén, L., and Olsson, P. O. Remarks on the variational method for scattering problems. Physical Rev. (2) 79, 532 (1950).

The relation between treatments of the variational method for scattering problems by Hulthén [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 14, no. 21 (1944); revised version, Ark. Mat. Astr. Fys. 35A, no. 25 (1948); these Rev. 6, 111; 10, 120] and by Huang [Physical Rev. (2) 76, 1878–1879 (1949); these Rev. 11, 464] is discussed. *C. Strachan* (Aberdeen).

Koch, O., Lesemann, K.-J., und Walther, A. Der radiale Temperaturverlauf im wandstabilisierten Quecksilber-Hochdruckbogen. Instrumentelle Integration der Elenbaas-Hellerschen Differentialgleichung unter Berücksichtigung der Temperaturabhängigkeit des Wärmeleitvermögens. Z. Physik 127, 153–162 (1950).

The authors deal with the "Elenbaas-Heller" differential equation for the temperature distribution in high voltage

mercury arcs. They generalise this equation and solve it numerically. Allowing for the dependence of the classical heat conductivity κ on temperature T in accordance with Sutherland's formula $\kappa=\kappa_0 T^\alpha$ (with $\alpha=.82$), they reach a nonlinear 2d order differential equation for the "reduced" temperature $\vartheta(\rho)$ as a function of the proportional distance $\rho=r/R$ (r=distance from axis, R=radius of cylindrical tube). The equation is

$$(1)\quad \frac{1}{\rho}\frac{d}{d\rho}\left\{\rho\vartheta^\alpha\frac{d\vartheta}{d\rho}\right\}+K_1\vartheta^1\exp\left\{-\frac{1}{2\vartheta}\right\}-K_2\vartheta^{-1}\exp\{-c\vartheta^{-1}\}=0$$

with constants K_1 and K_2 and c depending on the physical data. The boundary conditions for (1) are (2) $d\vartheta/d\rho=0$ at $\rho=0$, $\vartheta=\vartheta_w$ at $\rho=1$. With $y=c'\rho^2$ (1) is transformed into the form

$$(3)\quad \frac{d}{dy}\vartheta^{1+\alpha}=-\frac{1+\alpha}{y}\int\left\{g_1(\vartheta^{1+\alpha})-\frac{K_2}{K_1}g_2(\vartheta^{1+\alpha})\right\}dy.$$

For the numerical integration of (3) the differential analyser at the Darmstadt Institute of Applied Mathematics was used. Some technical details and keys for the set-up are given. For a number of ratios K_1/K_2 graphs for the solutions of the initial value problems $\vartheta(0)=\vartheta_0$, $d\vartheta/d\rho=0$ are produced from which those satisfying (2) are obtained by interpolation. *H. O. Hartley* (London).

Weizel, W., und Ecker, G. Einfluss des Eigenmagnetfeldes und der Diffusion auf eine wandstabilisierte Lichtbogensäule. Z. Physik **127**, 495–512 (1950).

The authors reexamine the physical foundations of, and modify terms in, the Elenbaas-Heller differential equation for the temperature (ϑ) distribution in a high voltage arc. The oversimplifications introduced by earlier workers to achieve analytic integration are dropped in view of the facilities offered by mechanical integration. Compared with previous generalisations of this equation [see, e.g., the preceding review] the main difference consists in a modified mechanism of the heat and energy convection (in which they drop the Sutherland correction in view of the high temperatures) which becomes essentially pressure-dependent. This makes it necessary to solve the initial value problem $\vartheta(0)=\vartheta_0$, $d\vartheta/d\rho\,|_{\rho=0}=0$ (ρ=proportional distance from axis) separately for any given combination of axial temperature ϑ_0 and pressure p. Introducing a temperature function of the form $G(\theta)=\int_{\vartheta_0}^{\vartheta}\phi(\tau;p)\,d\tau$ in place of the temperature θ as the dependent variable, the resulting equation becomes

$$F_1(G)+F_2(G)+\frac{d}{du}u\frac{dG}{du}=0,\qquad u=c\rho^2,$$

where $\phi(\theta)$, $F_1(G)$, and $F_2(G)$ are pressure-dependent functions depending on the physics of heat convection and obtained by a preliminary process of quadrature.

H. O. Hartley (London).

Byhovskiĭ, M. L. The accuracy of electrical circuits for calculation. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1239–1278 (1948). (Russian)

The parameters q_k of an electric circuit containing only resistances can be considered as generalized coordinates in terms of which, together with direct current voltage sources E_i, the resulting voltage at any point (output) is determined: $U=\sum E_i\varphi_i(q_1,\cdots,q_n)$. Errors in the q_k, called primary errors, and errors in the E_i cause an output voltage error ΔU. By expanding in Taylor's series and retaining the first order terms, ΔU is found to be the sum of three groups of terms: (1) the so-called output voltage variation error which is shown to equal the difference between the output voltage error in the final and initial condition (this is not the subject of further investigation in the paper); (2) the error arising from the primary errors $\sum E_j\sum(\partial\varphi_j/\partial q_k)\Delta q_k$; and (3) the so-called circuit errors which arise from errors in the E_j. In order to obtain an approximation to a typical term of the second group experimentally a transformed circuit is obtained from the given one by shortcircuiting the terminals containing the source E_j and introducing a pair of terminals for a complementary source at q_k. Four quantities measurable on the given and transformed circuit are needed to determine $\partial\varphi_j/\partial q_k$; technical details are explained. The methods are applied to a number of simple circuits by writing equations for the transformed circuit. Among these may be mentioned: (a) the error due to finite width of the potentiometer contact in the customary voltage divider arrangement for multiplication; and (b) apparatus for solution of linear equations, especially the circuit of H. Bode [Z. Angew. Math. Mech. **17**, 213–223 (1937)]. The method is extended to circuits involving reactive elements under steady state conditions by the use of complex representation and superposition. A circuit using a rectifier with negative exponential characteristic to produce the square of a voltage serves as an example of the discussion given for nonlinear elements. Finally, the expected value and standard deviation of the output voltage error is determined in terms of the expected value and standard deviation (Gaussian distribution) of primary errors. The concepts introduced and methods employed are frequently compared with the analogous treatment of the accuracy of mechanisms by N. G. Bruevič and his school [summarized in Bull. Acad. Sci. URSS. Cl. Sci. Tech. Izvestiya Akad. Nauk SSSR **1946**, 1065–1079; these Rev. **8**, 99]. *R. Church.*

Byhovskiĭ, M. L. The accuracy of electric networks intended for the solution of Laplace's equation. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 489–526 (1950). (Russian)

Using the assumptions of the paper reviewed above and the methods (especially the transformed circuit) introduced in it, the author here starts by studying the first order approximation to the error ΔU_i in the potential U_i between the ith and $(i+1)$st resistance of n series-connected resistances (a circuit for the one-dimensional Laplace equation) of conductance $g+\Delta g_k$ when the potential at one end is $U_0+\Delta U_0$ and at the other end is $U_n+\Delta U_n$. Assuming Gaussian distribution of the Δg_k with mean and standard deviation determined from the tolerance of the g's, the mean and standard deviation of ΔU_i due to these primary errors and also due to the inexact realization of the end conditions are determined. It may be observed that the standard deviation of the output error due to the primary errors increases as $n^{-\frac{1}{2}}$ and is greatest for $i=\frac{1}{2}n$ (n even) while the standard deviation of the error due to end conditions is greatest at the ends where it is $\sigma_{\Delta U_0}$. The discussion of the network corresponding to the two-dimensional Cartesian form of Laplace's equation is conducted along quite similar lines. The method of the transformed circuit reduces the determination of the partial derivatives needed to give the components of output error to the solution of a Poisson equation with zero boundary conditions. Results of experimental determination of these partial derivatives are given. It may be mentioned that the standard deviation of the error due to primary errors increases as $N^{-\frac{1}{2}}$, where N is the number of resistances. The standard deviation of the error

due to error in boundary conditions is discussed under the assumption that the errors at the network points corresponding to the boundary are (a) mutually independent or (b) equal in sets. The author explains how the network may be designed to attain prescribed accuracy. Errors due to solution of the boundary value problem by finite differences enter here but their study is not an objective of the paper.
R. Church (Annapolis, Md.).

Ryerson, Joseph L. The solution of differential equations by electrical analog computers. Amer. J. Phys. **19**, 90–97 (1951).

Kato, Tosio. On the upper and lower bounds of eigenvalues. J. Phys. Soc. Japan **4**, 334–339 (1949).

The author's purpose is to obtain the best possible lower bounds of eigenvalues by using quantities which are directly connected with the application of the Rayleigh-Ritz method (which gives upper bounds). He considers a self-adjoint operator H in a Hilbert space $\mathfrak{H}$ and gives sufficient conditions that a closed interval $[\alpha, \beta]$ contain at least m points of the spectrum. The most precise result is obtained for $m=1$ as follows: Using the quantity $(Hw, w)=\eta$ and $(Hw, Hw)=\eta^2+\epsilon^2$, where w is an arbitrary normalized element of $\mathfrak{H}$ (in the Rayleigh-Ritz method w represents a tentative approximation of an eigenfunction) he obtains that if $\alpha<\eta<\beta$ and $(\eta-\alpha)(\beta-\eta)=\epsilon^2$, the closed interval $[\alpha, \beta]$ contains at least one point of the spectrum. This result can be considered as a generalization of an estimate of a lower bound of the smallest eigenvalue given by G. Temple [see G. Temple and W. G. Bickley, Rayleigh's Principle and its Applications to Engineering . . . , Oxford, 1933]. The author remarks that the lower bounds obtained in this way are especially valuable when the eigenvalues are widely separated as is usually the case for ordinary differential equations, but in general will not give precise enough results for partial differential equations. The reviewer's only criticism is that in the introduction the author makes statements about "all" methods for the calculation of lower bounds. This may be misleading since his considerations only touch upon a special category of methods, and he is evidently unaware of the method introduced by A. Weinstein [Étude des spectres des équations aux derivées partielles . . . , Mémor. Sci. Math. no. 88, 1937; see also the reviewer, Proc. Nat. Acad. Sci. U.S.A. **34**, 594–601 (1948); these Rev. **10**, 382] which does not come under this particular category. *N. Aronszajn* (Stillwater, Okla.).

Curtis, A. R., Michel, J. G. L., and Sully, Elizabeth D. Central projection and stereoscopic diagrams on the differential analyser. Math. Gaz. **34**, 276–280 (1950).

Houbolt, John C. A recurrence matrix solution for the dynamic response of elastic aircraft. J. Aeronaut. Sci. **17**, 540–550 (1950).

This article gives a method to calculate the response of an aircraft to transient loads and special gusts. It is convenient to divide the problem into two phases, first the determination of the wing deflection under applied loads, second the determination of the loads from the dynamics of the airplane motion. For the first the wing semispan is divided into a number of sections with a station point at the center of each section. Then $A \cdot w=p$, where A is the flexibility matrix, w the column of the deflections, and p the column of the loads at the station points. Now the loads are the sums of inertia loads and aerodynamic loads, depending on w and its time derivatives. The derivatives are replaced by differences. At the start of the response the vertical displacements, velocities, and accelerations must be zero. Thus, the displacements at successive time points can be calculated and from these the loads and the stresses.
W. H. Muller (Amsterdam).

Samulon, H. A. Spectrum analysis of transient response curves. Proc. I.R.E. **39**, 175–186 (1951).

The paper describes a method of computation of amplitude and phase response of a network from its measured transient response. Tables and nomographs have been included to facilitate the numerical evaluation.
Author's summary.

Bowen, J. H., and Burnup, T. E. A simple analogue computor for Fourier analysis and synthesis. Electronic Engrg. **23**, 67–69 (1951).

Spring, Osc. W. Kleine Bemerkung zu einer Klasse versicherungstechnischer Approximationen. Mitt. Verein. Schweiz. Versich.-Math. **50**, 229–238 (1950).

The differential equation for annuities is used to show that certain approximation formulae used in a recent paper of H. Jecklin [same vol., 133–140 (1950); these Rev. **12**, 57] are closely related. Similar approximations are derived for more general actuarial functions and it is shown that the approximation is better the smaller the force of mortality.
E. Lukacs (Washington, D. C.).

Jecklin, H., und Zimmermann, H. Reserveberechnung auf Basis hyperbolischer Interpolation. Mitt. Verein. Schweiz. Versich.-Math. **50**, 179–196 (1950).

The author derives an interpolation formula for the premium-reserve ${}_tV$ by approximating the curve representing ${}_tV$ by an equilateral hyperbola. *E. Lukacs.*

ASTRONOMY

Ghosh, N. L. Spheroidal configuration under the law of density $\rho=\rho_0(1-\alpha r^2-\beta z^2)$. Bull. Calcutta Math. Soc. **42**, 101–117 (1950).

This paper investigates possible figures of a fluid mass in a steady rotation when the equations of motion of the fluid alone are taken into account and no physical connection between $\bar{p}$ and $\bar{\rho}$ is assumed. On postulating a spheroidal distribution of the density, the pressure and the angular velocity are chosen in a form satisfying the equations of motion. The function $\bar{\omega}^2/2\bar{a}\bar{\rho}$ ($\bar{\omega}$, $\bar{\rho}$ average values), the configurations for variable ρ_0, $\bar{\omega}$ and a, and the energy of the model are also investigated. *W. S. Jardetzky.*

Danjon, André. Adaptation de la méthode de Laplace aux approximations successives. C. R. Acad. Sci. Paris **231**, 673–676 (1950).

The author proposes to compensate for the suppression of terms of order greater than the second in the method of Laplace using three observations by adding algebraically the differences between the observed and calculated coordinates to the former and using these new values in place of the observed for a recomputation, the process to be repeated for higher orders of approximation. This orbit is then to be corrected by a variational scheme by which the residuals at all observations available are made to vanish. A

paper giving details of the process is to appear in another journal. *R. G. Langebartel* (Saltsjöbaden).

Samollova-Yahontova, N. S. On the correction of the elements of elliptic orbits. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. 4, no. 6(59), 255–264 (1949). (Russian)

The author examines various methods of orbit correction to determine their relative advantages and disadvantages in practical work. The equations of condition for each scheme are all put in the same general form, the right members being in each case the product of two cracovians. From a study of these cracovians the author concludes that the methods of Eckert and Brouwer [Astr. J. 46, 125–132 (1937)] and Banachiewicz [Acta Astr. Sér. c. 1, 71–86 (1929)] are preferable. *R. G. Langebartel* (Saltsjöbaden).

Lancelin, F. Utilisation des variables canoniques de Poincaré. Contribution à l'étude des planètes troyennes. Bull. Astr. (2) 13, 107–121 (1948).

Using the canonical variables of Poincaré [Leçons de mécanique céleste, vol. 1, Gauthier-Villars, Paris, 1905, p. 80], this posthumous paper develops the generating function for the case of the Trojan asteroids up to the fourth power terms in the eccentricity of the asteroid's orbit.

R. G. Langebartel (Saltsjöbaden).

Roure, Henri. Théorie nouvelle des grosses planètes du système solaire. Bull. Astr. (2) 15, 1–103 (1950).

In introductory paragraphs the author states that the purpose of his work is the development of theories of the motions of principal planets especially with the view of constructing convenient tables. The method presented is essentially that introduced by G. W. Hill in the lunar theory, more especially in the form presented by H. Poincaré [Leçons de mécanique céleste, vol. 2_2, Gauthier-Villars, Paris, 1909]. The equations of motion for each planet are treated first on the assumption that the orbits of the other planets are unperturbed ellipses. The inclinations are taken into account in the first approximations. The form of solution permits the introduction of the motions of the perihelia and nodes in the same form as in the lunar theory, i.e., these motions are not permitted to produce the pure or mixed secular terms present in the common types of planetary theory in which the time occurs explicitly in the coefficients of trigonometric terms. A novel feature of the theory is that the disturbing function is developed in precisely the same form as in the lunar theory, i.e., in powers of the ratio of the semi-major axes of the planets. In the lunar theory this ratio is 1:400, and thus assures rapid convergence. In the planetary system this ratio is much less favorable, exceeding 0.7 for Venus-Earth and Neptune-Pluto. The method is presented clearly and elegantly, and the author gives in considerable detail the process that he proposes to use in successive approximations. In the opinion of the reviewer two important questions that may determine the value of the method remain to be established: (1) While the proposed method of development of the disturbing function may be permissible, the method may be too slowly convergent numerically to be of practical value. (2) Eventually the provisional solutions for the motions of the perihelia and the nodes must be replaced by a general solution that takes into account the interactions of these motions among the several planets. The analytical and practical difficulty of this aspect of the problem is not considered. *D. Brouwer* (New Haven, Conn.).

Radzievskiĭ, V. V. The restricted problem of three bodies taking account of light pressure. Akad. Nauk SSSR. Astr. Žurnal 27, 250–256 (1950). (Russian)

The author discusses in a qualitative manner the dependence of the position of libration points on the intensity of light pressure from the main body when that force is added in the restricted problem of three bodies.

R. G. Langebartel (Saltsjöbaden).

Motz, Lloyd. The apsidal motion of giant binary stars. Astrophys. J. 112, 434–444 (1950).

The theory of distorted polytropes is applied to two different models of giant stars, each consisting of a convective core, an intermediate zone, and a radiative envelope. The apsidal motion coefficients obtained yield values for the effective polytropic index of each of the three-zone models investigated. One of the two results is in excellent agreement with the observational determination for the only giant binary system for which a fairly reliable observed apsidal motion is available. *D. Brouwer.*

Epstein, I. Pulsation properties of giant-star models. Astrophys. J. 112, 6–21 (1950).

The question has been discussed whether the energy-producing mechanism in the red giant stars could be the normal carbon cycle. For this to be possible the giant stars must have very high central concentrations. From integration on pure polytropics one would expect a decrease in pulsation period with increasing central concentration. If this is so, high central concentration in giants would give very short periods which would again mean a discrepancy between theory and observation. In the present paper the author solves the pulsation equation for the fundamental and first over-tone mode for three giant star models with high central concentration. To obtain agreement between theory and observation with regard to the period-period density relationship it would be necessary to reduce the cepheid masses deduced from the mass-luminosity relation by a factor of four, or to reduce the effective temperature estimated from the colours of cepheids by about 13%, or lengthening the theoretical periods by assuming a very deep hydrogen convection layer. *G. Randers* (Oslo).

Gething, P. J. D. A critical examination of the gravitational theory of the origin of cosmic rays. Proc. Roy. Soc. London. Ser. A. 204, 278–294 (1950).

The author gives a new derivation of Milne's theory of the origin of cosmic rays which is based on kinematic relativity. The main innovation is the use by the author of a different time scale. He states that Milne's theory agrees with all known data on cosmic rays. *A. Schild.*

Vaidya, P. C. A radiation-absorbing centre in a non-statical homogeneous universe. Nature 166, 565 (1950).

The author exhibits a spherically symmetric relativistic line element which does not become flat at large distance. It is hoped that this solution might serve as a model for a "dark nebula" which absorbs radiation from the surrounding space. *A. Schild* (Pittsburgh, Pa.).

Rutgers, A. J. Is matter being continually created? Nederl. Tijdschr. Natuurkunde 16, 161–170 (1950). (Dutch)

The author presents a survey of cosmological theories, including the older work by Einstein, de Sitter, Friedmann, and Lemaître and the more recent developments, especially by F. Hoyle and P. Jordan. *D. Brouwer.*

Bleksley, A. E. H. A new approach to cosmology. I. South African J. Sci. **46**, 343–346 (1950).

Bleksley, A. E. H. A new approach to cosmology. II. South African J. Sci. **47**, 13–15 (1950).

MECHANICS

*Beghin, H., et Julia, G. **Exercices de mécanique.** Tome I. Fasc. 1. 2d ed. Gauthier-Villars, Paris, 1946. vii + pp. 1–338.

*Beghin, H., et Julia, G. **Exercices de mécanique.** Tome I. Fasc. 2. 2d ed. Gauthier-Villars, Paris, 1951. i+pp. 339–583.

The first edition appeared in 1931.

de Weert, Th. L. On inertial forces. Simon Stevin **27**, 136–141 (1950). (Dutch)

A point mass free to slide on a rod and rotating horizontally is attached to a string led, parallel to the rod, to the axis of rotation, where it is deflected downward and supports another mass. The horizontal rod is set rotating at a certain angular speed and at time zero the masses are released. The author calculates the work necessary over a period of time to counteract the retarding (or accelerating) torque induced by the Coriolis force, keeping the angular velocity constant. Friction effects are neglected.

W. J. Nemerever (Stillwater, Okla.).

Peyroux, J. Calcul des pendules et des tensions de lignes souples à suspension caténaire. Rev. Gén. Électricité **59**, 553–561 (1950).

Haag, Jules. Sur le coefficient de restitution dans la théorie des chocs. C. R. Acad. Sci. Paris **231**, 257–259 (1950).

Assuming an elastic model, the author studies the dependence of the coefficient of restitution on the velocity.

P. Franklin (Cambridge, Mass.).

Nagendra Nath, N. S., and Kumar Roy, Sanat. Vibrations of an infinite linear lattice consisting of two types of particles. II. Proc. Indian Acad. Sci., Sect. A. **31**, 330–338 (1950).

[For part I see same Proc., Sect. A. **28**, 289–295 (1948); these Rev. **10**, 489.] For arbitrary small initial displacements and velocities, asymptotic expressions for the state of vibrations of the particles of an infinite linear lattice are obtained.

P. Franklin (Cambridge, Mass.).

Hohlov, A. F. On the use of a gyromagnet as a course indicator. Avtomatika i Telemehanika **8**, 285–296 (1947). (Russian)

The fact that for aircraft flying at large speeds and in particular when heading West at high latitudes φ the aircraft speed may reduce to zero the direction indicating force of the gyrocompass (gyropendulum) seems to have precluded its use in aircraft. The author is concerned with the solution of the problem of finding a course indicator for which the direction indicating force contains a component independent of the speed of the aircraft. It turns out that the solution of this problem is possible to some extent if instead of one pendulous moment two magnetic moments of the geomagnetic field are applied to an astatic gyroscope with three degrees of freedom. The results of the paper may be summarized as follows: (1) Under a certain condition which can always be realized technically the gyromagnet stabilizes in the plane of the horizon between the geographic and the magnetic meridional planes; if, in addition, the condition $\varphi=\mu$ (μ magnetic inclination) is satisfied, then stabilization occurs just in the middle meridional plane between the above-mentioned planes. The period of the undamped precessional vibrations equals one-half of the period of rotation of the earth. (2) The direction indicating force of a gyromagnet is almost twice that of the direction indicating force of a gyrocompass (for small declinations). (3) Course deviations in azimuth of a gyromagnet are twice as small as the corresponding deviations of a gyrocompass. There are no course deviations of a gyromagnet in the vertical plane. (4) Because of the large period of vibrations the ballistic deviations of a gyromagnet are negligibly small, and the gyromagnet goes over to new positions of equilibrium only after a certain lapse of time. A numerical example is given.

E. Leimanis (Vancouver, B. C.).

Agostinelli, C. Sul problema dei tre corpi. Matematiche, Catania **5**, 45–65 (1950).

Expository lecture.

Hydrodynamics, Aerodynamics, Acoustics

Truesdell, C. On the balance between deformation and rotation in the motion of a continuous medium. J. Washington Acad. Sci. **40**, 313–317 (1950).

This paper is a purely kinematic study of fluid motion with emphasis on two scalars: the magnitude w of the vorticity vector $w_i=-e_{ijk}v_{j,k}$, where v_j is the velocity, and an invariant of deformation $\Pi_d=\frac{1}{2}e_{ijk}e_{lmn}d_{jl}d_{km}=\frac{1}{2}(d_{jl}d_{mm}-d_{jm}d_{jm})$, where e is the permutation symbol and $2d_{jk}=v_{j,k}+v_{k,j}$. The author shows that if the container is fixed and the fluid adheres to it (with suitable conditions at infinity), then there is a balance between vorticity and deformation in the sense that $\int(4\Pi_d+w^2)\,dV=0$. He also discusses motions for which the balance holds at each point ($4\Pi_d+w^2=0$) and generalises a theorem of Bilimovitch [Acad. Serbe Sci. Publ. Inst. Math. **2**, 37–52 (1948); these Rev. **10**, 517; the omission of a factor 4 in that review is noted]. This involves consideration of the velocital volume V_V and the hodograph volume V_H of an arbitrary region V bounded by a surface S: V_V is the volume enclosed by the surface formed by the extremities of the velocity vectors drawn from the points of S, and V_H the volume formed by drawing these velocity vectors from a fixed point. The dimensional heterogeneity involved in the definitions of these volumes is noted and used.

J. L. Synge (College Park, Md.).

Etkin, Bernard, and Szebehely, Victor G. Comments on Truesdell's paper on Bernoulli's theorem for viscous compressible fluids. Physical Review (2) **80**, 767 (1950).

The reviewer [same Rev. (2) **77**, 535–536 (1950); these Rev. **11**, 472] proved the following theorem: Let the curves C be everywhere tangent to the plane of vorticity and velocity and everywhere normal to the curl of the vorticity; then along these curves, in any flow of a viscous incompressible fluid of uniform viscosity, if the vorticity be steady and the extraneous force be conservative, Bernoulli's theorem in the classical form remains valid. The authors discuss

a more general result of the same type given by the reviewer [loc. cit.]. They find some alternatives, a simpler result gained from stronger assumptions, one redundancy in a special case, an absence of applications, and three misprints.

C. Truesdell (Bloomington, Ind.).

Booth, F. The cataphoresis of spherical, solid non-conducting particles in a symmetrical electrolyte. Proc. Roy. Soc. London. Ser. A. **203**, 514–533 (1950).

In this paper, a general method is given for calculating the cataphoretic velocity of a charged sphere suspended in a viscous liquid at rest under the action of the electrostatic field. The assumptions made are: (1) the dielectric constant, the viscosity and the ionic mobilities are uniform throughout the fluid phase; (2) the cataphoretic velocity is proportional to the field strength; (3) the electrolyte is incompressible; (4) the inertia terms are small; (5) the electrolyte contains equal numbers of ions of equal but opposite charge; (6) (a) the thickness of surface phase is small compared with particle radius; (b) the surface charge is immobile; (c) the surface charge densities with and without applied field are superposable; (d) the potential difference across the surface phase retains its equilibrium when external field is applied. The author carries out the solutions to the second approximation. The results for the cataphoretic velocity are compared with that of several previous authors and good agreement is obtained only with that of Overbeck [Kolloid-Beihefte **54**, 287–364 (1943)].

Y. H. Kuo.

Booth, F. The electroviscous effect for suspensions of solid spherical particles. Proc. Roy. Soc. London. Ser. A. **203**, 533–551 (1950).

Following the general procedure of the paper reviewed above, the author makes an extended study after Smoluchowski [Kolloid-Z. **18**, 190–195 (1916)] on the electroviscous effect due to the charged particles suspended in an electrolyte. Exactly the same assumptions were made as in the preceding paper and, in addition, the author assumed that the thickness of the double layer is small compared with the average distance between the neighboring particles and that the particles are nonconducting. The final result for the effective viscosity η is found to be

$$\eta=\eta_0\left\{1+\frac{5}{2}\left(\frac{v}{V}\right)\left(1+\sum_2^{\infty}a_rQ^r\right)\right\},$$

where η_0 is the viscosity of the electrolyte, v the volume suspended in volume V of solution, and Qe is the charge on each particle. The coefficient is evaluated and the result agrees fairly well with the observed values.

Y. H. Kuo (Ithaca, N. Y.).

Tetervin, Neal, and Lin, Chia Chiao. A general integral form of the boundary-layer equation for incompressible flow with an application to the calculation of the separation point of turbulent boundary layers. · Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2158**, 63 pp. (1950).

The equation of longitudinal momentum for a laminar or turbulent boundary layer in two-dimensional or axisymmetrical flow is multiplied by the mth power of the longitudinal velocity component u and by the nth power of the normal distance y from the solid surface and integrated across the boundary layer. Special cases are the integral equations of momentum, of energy, and of moment of momentum about an axis along the surface. The general equation relates the development of shape parameter H and momentum thickness θ to the known development of main stream velocity U and the unknown distribution of shearing stress in the boundary layer. The equation for $m=n=0$ (the momentum integral equation) is subtracted at this stage to eliminate $d\theta/dx$. The remaining equation is then specialised, for the case of turbulent boundary layers, by assuming that the velocity profile depends only on the single parameter H. A further specialisation, in which the velocity u is proportional to the $\frac{1}{7}(H-1)$th power of y, is made; this is said to be justified by experiment when $H<1.8$. There is a general discussion of the high potential utility of the results in this final form when combined with the right experimental data. In some preliminary attempts at applying the equation (with m, n given particular values, not always the same in each attempt) three postulates concerning shearing stress distribution were tried. (i) It depends only on the Pohlhausen parameter. (ii) It is uniform across the boundary layer. (iii) It depends only on the shape parameter H. Since the predicted development of H is not in very good agreement with experiment on any of these hypotheses, it is inferred that none of them are sufficiently accurate.

M. J. Lighthill (Manchester).

Tani, Itiro. On the solution of the laminar boundary layer equations. J. Phys. Soc. Japan **4**, 149–154 (1949).

The author has extended Howarth's method of approximate solution for laminar boundary layer flow with a linear velocity distribution outside the boundary layer to the case where the velocity is given by $U=U_0+cx^n$. Here $n=2, 4, 8$; U_0, c are constants, and x is the coordinate measured along the surface. The velocity profiles were first calculated exactly by power series and then approximately by Pohlhausen's method. In the second step, the author introduced a new parameter $\omega=\sigma^2U(d^2U/dx^2)/(dU/dx)^2$ in addition to $\sigma=(\theta^2/\nu)(dU/dx)$ as was generally used, where θ is the momentum thickness and ν the kinematic viscosity coefficient. By successive approximation, the skin friction and the velocity profile were calculated. The author claims that the separation point determined by his method is considerably later than that found previously.

Y. H. Kuo.

Tani, Itiro. Water jump in the boundary layer. J. Phys. Soc. Japan **4**, 212–215 (1949).

Approximate solution of the equations for the flow from a point source in a horizontal plane with the boundary layer approximation at the plane, a free surface above the plane, and gravity acting.

J. V. Wehausen.

Emmons, H. W. Note on aerodynamic heating. Quart. Appl. Math. **8**, 402–405 (1951).

In the established solution of the infinite flat plate boundary layer starting from rest the effect of viscous energy dissipation has been neglected. This effect is considered in detail here with the laminar temperature profile obtained for arbitrary Prandtl number. Explicit results in integral form are given for both the adiabatic and conducting wall. In particular, the temperature recovery factor is shown to be $r=2[\mathrm{Pr}/(2-\mathrm{Pr})]^{\frac{1}{2}}[1-(2/\pi)\sin^{-1}(\mathrm{Pr}/2)^{\frac{1}{2}}]$. It is shown that the transient profile is similar to the steady state profile for a semi-infinite plate with the position of similarity at a distance from the leading edge equal to $\frac{1}{3}U_0t$ for velocity and $\frac{1}{2}U_0t$ for temperature.

N. A. Hall.

Meksyn, D. Stability of viscous flow over concave cylindrical surfaces. Proc. Roy. Soc. London. Ser. A. **203**, 253–265 (1950).

The author studies the stability of viscous flow over concave cylindrical surfaces with respect to a kind of disturbance considered by Görtler. Asymptotic integration of the differential equation is used instead of the integral equation method used by Görtler, the latter method being convenient only for the lowest mode. The results agree qualitatively with Görtler's but with somewhat different numerical values for the critical Reynolds number. It is further shown that there is a finite number of steady states for each value of the Reynolds number. *C. C. Lin.*

Batchelor, G. K. Diffusion in a field of homogeneous turbulence. I. Eulerian analysis. Australian J. Sci. Research. Ser. A. **2**, 437–450 (1949).

General remarks, mostly of an expository nature, on the mathematical description of the diffusion of some property by a turbulent flow when an initial distribution of the property is given. *J. V. Wehausen* (Providence, R. I.).

Kaplan, Carl. On the particular integrals of the Prandtl-Busemann iteration equations for the flow of a compressible fluid. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2159**, 16 pp. (1950).

The Prandtl-Busemann iteration equations are those which treat the irrotational flow of a gas by a system of successive approximations based on the supposition that the disturbances to the uniform main stream are not large. The first approximation is that usually associated with Prandtl and Glauert, equivalent to the theory of sound in a frame of reference moving with the main stream. Higher approximations retain squares, cubes, etc. of disturbances successively; the solutions already obtained being substituted in the nonlinear terms, leaving a nonhomogeneous linear equation to be solved for the next approximation. In this paper the author sets up the equations in terms of the velocity potential, and uses the complex variable method to find (given the solution on the first approximation) particular integrals for the equations for the second and third approximations. In the light of this work it is now only necessary, in a given problem, to add on a solution of the Prandtl-Glauert equation so that the whole second or third approximation satisfies the boundary conditions. [Reviewer's note. It would be useful if someone would do the same work on the equations for successive approximation to the stream function. It appears from the work of Hantzsche and Wendt [Z. Angew. Math. Mech. **22**, 72–86 (1942); these Rev. **4**, 177] that the boundary conditions can be satisfied more easily when the stream function is used.]

M. J. Lighthill (Manchester).

Harder, Keith C., and Klunker, E. B. On a source-sink method for the solution of the Prandtl-Busemann iteration equations in two-dimensional compressible flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2253**, 10 pp. (1950).

Bernard, J. J., et Siestrunck, R. Sur l'utilisation d'approximations successives dans la détermination de certains potentiels aérodynamiques. Recherche Aéronautique **1950**, no. 17, 3–8 (1950).

In the solution of certain potential problems of the third kind by methods of electrical analogy, one may encounter terms of such sign that "negative resistance" is required. The present paper presents methods of successive approximation intended to avoid this difficulty; the successive steps can be solved by ordinary methods. The general case is included, where the negative values may occur only on part of the boundary. As examples, the authors consider the lifting-line problem of Prandtl, wherein negative lift slopes may be introduced for special purposes, and also a problem arising in the study of dual-rotation propellers.

W. R. Sears (Ithaca, N. Y.).

Bergman, S. Determination of axially symmetric flow patterns of a compressible fluid. J. Math. Physics **29**, 133–145 (1950).

The equation for the Legendre potential in terms of the velocity components in compressible irrotational axisymmetric flow is written with linear terms (those appropriate to two-dimensional flow) on the left and the remaining nonlinear terms on the right. The naturally resulting sequence of successive approximations to axisymmetrical flow, starting from the analogous two-dimensional flow as a first approximation, is then studied in some detail.

M. J. Lighthill (Manchester).

Klein, Milton M., and Perl, W. Calculation of compressible potential flow past slender bodies of revolution by an integral method. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2245**, 54 pp. (1950).

Clauser, Francis H. Two-dimensional compressible flows having arbitrarily specified pressure distributions for gases with gamma equal to minus one. Symposium on theoretical compressible flow, 28 June 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1132**, pp. 1–33 (1950).

Mangler [Jahrbuch Deutsch. Luftfahrtforschung **1938**, 46–53] gave a procedure for aerodynamic design by computing the shape of solid surfaces such that the exact potential flow past them should have a prescribed distribution of surface velocity as a function of distance along the surface. However, this problem has in general no real solution, and the calculated flow is on a Riemann surface with an infinity of sheets. Hence the reviewer [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2112** (1945)] advanced an improved method in which the velocity distribution was presented on a circle, such that the flow field was imagined transformed into the domain outside it. Then the conditions for existence may be imposed in advance, while the velocity-distance relation may also be checked in advance (for practical utility), with little labour.

The present paper extends this work to general subsonic flow, under the Kármán-Tsien approximation. Thus it enables aerofoils to be designed with a given surface pressure distribution at a given high subsonic speed. Aerofoils with slots, slats, and other ducting, and aerofoils in cascade, are also treated, following discussions by many authors for the case of zero Mach number. However, there is no explicit statement that the calculation has been carried out for any particular case. In its absence one can only assume that the many illustrations in the text represent conjectural, not calculated, shapes.

M. J. Lighthill (Manchester).

Heaslet, Max. A., Lomax, Harvard, and Spreiter, John R. Linearized compressible-flow theory for sonic flight speeds. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 956, 16 pp. (1950).

A very thorough account, including a critical survey, and an account of calculation methods and results, is given of the use of the linearised equation of gas flow for three-dimensional steady problems and two-dimensional unsteady problems at Mach number 1. The work is considered to be justified when the calculated surface disturbances are small, since it is then at least self-consistent. In the steady case this is so only for wings of zero thickness (lifting surfaces); the data given under this condition are very extensive.

M. J. Lighthill (Manchester).

Germain, Paul. La théorie générale des mouvements coniques et ses applications à l'aérodynamique supersonique. O.N.E.R.A. Publ. no. 34, viii+197 pp. (1949).

In the study of small-perturbation flows at supersonic speeds there appear cases of conical flow, i.e., velocity components have the form $F(x_2/x_1, x_3/x_1)$ or $f(\theta, \chi)$, where x_1, x_2, x_3 are Cartesian coordinates; x_1 in the stream direction, $x_2=r\cos\theta$, $x_3=r\sin\theta$, $\chi=x_1/\beta r$; $\beta^2=M^2-1$; and M is the stream Mach number. The author obtains the differential equation for flow of this type in the forms: for $\chi>1$: $f_{\xi\xi}+f_{\eta\eta}=0$, where $\chi=\cosh\xi$; for $\chi<1$: $f_{\eta\eta}-f_{\eta\eta}=0$, where $\chi=\cos\eta$. He proposes to study two main types of flow: flow past slender cones; and flow past nearly flat cones (conical wings). He also points out the more general cases called "homogeneous" flows, i.e., those for which the velocity potential is $\varphi=r^n f(\theta, \chi)$, n being a nonnegative integer.

Cases involving slender cones are treated by function-theoretical methods; application is made to circular and elliptic cones, a cone whose section has a sharp corner, a slender delta wing, and a cone of semicircular section. General formulas for lift and drag are found. The problem of numerical calculation is treated in considerable detail. Passing on to conical wings, the author treats separately the cases of supersonic and subsonic leading edges. Again, the mathematical techniques are those of complex-variable theory. Electrical-analog methods are explained and have been used to obtain some numerical results. In the final chapter, it is shown how conical flows can be used to build up flow patterns over other configurations. These include rectangular and swept-back wings, ailerons, wings with dihedral, and crossed wings.

The main part of the report was prepared in 1948 and makes reference to much of the pertinent literature available at that time. There are six appendices, added to the report in 1949, giving further development and applications, and referring to some more recent publications.

W. R. Sears (Ithaca, N. Y.).

Mirels, Harold. Lift-cancellation technique in linearized supersonic-wing theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2145, 43 pp. (1950).

The idea of the method, which has been used by Goodman [J. Aeronaut. Sci. 16, 365–374, 703–704 (1949); these Rev. 11, 224] and others, is reviewed. It consists of replacing a given wing by a simpler one of larger extent, such as a wing of infinite span, and a superimposed "cancellation wing" which corrects the boundary condition. It is shown how the load distribution in the overlapping region can be calculated from the known part of the cancellation wing. The mathematical method utilizes Hadamard's "finite part" of infinite integrals and involves, basically, inversion of an integral equation of Abel's type. If the given wing has a subsonic trailing edge, there appears an indeterminacy which permits satisfaction of the Kutta condition. The method is applied to a number of practical examples, including curved planforms and both leading- and trailing-edge cancellations, and one case of a reentrant side edge. The paper under review affords some simplification and generalization of Goodman's work [loc. cit.].

W. R. Sears (Ithaca, N. Y.).

Ribner, Herbert S. Some conical and quasi-conical flows in linearized supersonic-wing theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2147, 61 pp. (1950).

It is proposed to apply the "cancellation" method described in the preceding review in an approximate form to some cases of wings having subsonic trailing edges. The approximation is to consist of taking the lift distribution in the central portion of a flat delta wing to be uniform; this was previously suggested by Cohen [same Tech. Notes, no. 1555 (1948); these Rev. 9, 478]. The three basic cancellation flows used in this approximation to produce swept-back wings at incidence, rolling, and pitching are derived here. These problems lead to integral equations, which are solved by considering the conical or quasi-conical natures of the flows. The solutions appear to be nonunique, and the choice of the physically most plausible solution is made by comparison with certain closely related cases where the correct solution can be found. The results are applied to practical cases in a subsequent report [ibid., no. 2146 (1950)]. In the concluding paragraphs, reference is made to Mirels' and Goodman's papers [see the preceding review and reference cited there]. The Goodman-Mirels method is considered in some detail in an appendix. Another appendix, contributed by F. K. Moore, is devoted to the differentiation with respect to a parameter, of singular (but integrable) definite integrals.

W. R. Sears (Ithaca, N. Y.).

Beane, Beverly. The characteristics of supersonic wings having biconvex sections. J. Aeronaut. Sci. 18, 7–20 (1951).

Robinson, A. Aerofoil theory for swallow tail wings of small aspect ratio. Coll. Aeronaut. Cranfield. Rep. no. 41, 16 pp. (2 plates) (1950).

This is an application of the slender-wing theory of Munk, Jones [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 835 (1946); these Rev. 11, 698] and others to highly sweptback wings with pointed tips. As usual, the only difficulty arises in satisfying the boundary and trailing-edge conditions behind the root-chord trailing edge. In this analysis an integral equation of Volterra's type appears, and is solved by an approximate numerical method. Lift, pitching moment, and chordwise distribution of load are calculated. The drag is also found by a procedure that involves a limiting process to evaluate the leading-edge suction. In a corrigendum dated January 1951 an error in the equation for the potential is pointed out, which invalidates much of the detailed analysis of the report. A new table of numerical results is also given, and it is stated that the analysis can be carried on by the same methods as before and will be given in detail in a revised report.

W. R. Sears (Ithaca, N. Y.).

Küchemann, D., and Weber, J. On the chordwise lift distribution at the centre of swept wings. Aeronaut. Quart. 2, 146–155 (1950).

This is an attempt to determine the effect of sweepback on the pressures on a lifting surface near its center (the "kink"). The authors begin with a continuous distribution of vortices $\gamma(x)$ in the shape of a sweptback constant-chord infinite wing. As is well known, this leads to logarithmically infinite induced velocities at the kink. They suggest, on the basis of some speculation, the use of the following form as an approximation:

$$(1) \qquad v(x) = -(2\pi)^{-1}\left\{\int_{-1}^{1}\gamma(\xi)(x-\xi)^{-1}d\xi+\sigma\gamma(x)\right\},$$

where $v(x)$ is the downwash at the center of the wing, $x=\pm 1$ denote the leading and trailing edges there, and σ depends on the sweep angle ϕ. For constant $v(x)$, (1) would admit solutions of the form $\gamma(x)=c[(1+x)/(1-x)]^n$, but the various constants would remain undetermined. The authors nevertheless decide to adopt this form and to determine n from experimental data concerning the aerodynamic-center locations of kink sections. Some data are shown, and it is concluded that $n=\frac{1}{2}-\phi/\pi$. It is shown that distributions having this exponent can be fitted roughly to experimental pressure distributions for the kink sections of 45° and 59° sweptback wings. *W. R. Sears* (Ithaca, N. Y.).

Neumark, S. Critical Mach numbers for swept-back wings. Aeronaut. Quart. 2, 85–110 (1950).

The method of source-sink distribution is used to calculate, to the first order, the velocity and pressure distributions on symmetrical sweptback wings in incompressible inviscid flow at zero incidence. The pressure distribution on a constant-chord sweptback wing of infinite span is presented in formulas and graphs, but details of the calculation are not shown. The minimum pressure occurs near the center section ("kink"). It is pointed out that this problem was attacked by H. Ludwieg by the same theory [Ministry of Aircraft Production, Völkenrode, Reports and Translations no. 84 (1946)] but that a mathematical error led him to a false conclusion. The present result is next generalized to apply to any profile whose plane-flow characteristics are known. Some results are also presented for finite-span sweptback wing and for tapered sweptback wings, but again the details of the calculation are withheld. Finally, the critical Mach numbers of such wings are calculated according to the Prandtl-Glauert approximation. In contrast to the outboard part of the wing where "simple sweep" theory applies, the sonic velocity always occurs near the kink at a subsonic Mach number. It appears to the reviewer that this work is related to that reported in detail by Ursell [same Quart. 1, 101–122 (1949)], but no reference is made. Ursell also discussed the mathematical error of Ludwieg and referred to a paper by the author on this same subject [Royal Aircraft Establishment Report no. Aero. 2200 (1947)]. *W. R. Sears* (Ithaca, N. Y.).

Lomax, Harvard, Sluder, Loma, and Heaslet, Max. A. The calculation of downwash behind supersonic wings with an application to triangular plan forms. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 957, 19 pp. (1950).

The material given is a combination of two previously published papers [Heaslet and Lomax, Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1620 (1948); Lomax and Sluder, ibid., no. 1803 (1949)]. *M. J. Lighthill* (Manchester).

Malvestuto, Frank S., Jr., Margolis, Kenneth, and Ribner, Herbert S. Theoretical lift and damping in roll at supersonic speeds of thin sweptback tapered wings with streamwise tips, subsonic leading edges, and supersonic trailing edges. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 970, 16 pp. (1950).

Stewartson, K. On the linearized potential theory of unsteady supersonic motion. Quart. J. Mech. Appl. Math. 3, 182–199 (1950).

The author considers the solution of the linearized equation (1) $c^2\nabla^2\psi = V^2\psi_{zz}+2i\omega V\psi_z-\omega^2\psi$ for harmonic motion of a thin wing in an idealized, supersonic flow by effecting a Laplace transformation in the streamwise (z) direction and, for three-dimensional cases, a Fourier transformation in the spanwise direction. The specific problems treated are the two-dimensional wing, the wide and narrow delta wings, the semi-infinite, rectangular wing, and the slender body of revolution. The potential for the wide delta wing is presented in integral form, and it is shown that the lift and moment coefficients are independent of sweepback, in agreement with previous results [J. W. Miles, J. Aeronaut. Sci. 16, 568–569 (1949); these Rev. 11, 273]. The solution for the rectangular wing is obtained by analogy to the classical Sommerfeld half-plane problem [H. Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, p. 538] and is in agreement with that obtained by the reviewer [J. Aeronaut. Sci. 16, 381 (1949); these Rev. 10, 755] but not with that of Stewart and Li [see the following review] in giving a tip correction containing only trigonometric (and not Bessel) functions. The solution for the narrow delta wing is obtained by expanding in powers of the frequency, a method that appears to be more amenable to computation than the expansion in orthogonal functions proposed by Robinson [Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 500–514; these Rev. 11, 477] and Haskind and Falkovich [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 371–376 (1947); translated as Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1257 (1950); these Rev. 9, 114; 11, 754] and the related expansion proposed by Germain and Bader [Recherche Aéronautique 1949, no. 11, 3–13; these Rev. 11, 476]. Numerical coefficients are given for the first three terms in the expansion of χ in powers of ω^2, where χ differs from ψ by an exponential factor and satisfies an equation of the type (2) $\chi_{xx}+\chi_{yy}=\alpha^2(\chi_{zz}+\lambda^2\chi)$. It is indicated that this same method may be generalized to treat the case of a semi-infinite swept wing, although not without considerable difficulty. The author also includes a brief indication of the formulation of the integral equation for more general wing problems. *J. W. Miles* (Auckland).

Stewart, H. J., and Li, Ting-Yi. Periodic motions of a rectangular wing moving at supersonic speed. J. Aeronaut. Sci. 17, 529–539 (1950).

The authors use an unpublished extension of Evvard's work [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1382 (1947); these Rev. 8, 610] to solve the problem of the oscillating, supersonic, rectangular wing. The same problem has been solved elsewhere by three different methods [J. W. Miles, same J. 16, 381 (1949); 17, 64 (1950); these Rev. 10, 755; 11, 476; the preceding review], the analytical results of which are in mutual agreement and in disagreement with the present results, albeit numerical results appear to exhibit reasonable agreement at several points. The development of

the method is not given, but the reviewer has discussed it with the authors, who now believe that it may be in error and are reconsidering the derivation. *J. W. Miles.*

Evvard, John C. Use of source distributions for evaluating theoretical aerodynamics of thin finite wings at supersonic speeds. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 951, 32 pp. (1950).

The "integral equation" method, which the author has used in numerous publications to solve problems of steady and unsteady supersonic flow past wings on the linearised theory, is expounded in a unified and extended form. A large number of figures are given illustrating the results obtained by applying the method to particular practical cases.

M. J. Lighthill (Manchester).

Souriau, Jean-Marie, et Chastenet de Géry, Jérôme. Extension de la méthode de Küssner aux profils épais. Recherche Aéronautique 1950, no. 17, 9–15 (1950).

Consider a profile in unsteady motion in an incompressible ideal fluid. Let φ_0, ψ_0 be introduced as coordinates, where $\varphi_0(x, y)$, $\psi_0(x, y)$ denote the velocity potential and stream function for steady flow past the same profile in its mean position. The boundary condition at the surface is formulated in these terms and is linearized assuming small deviations from the mean position. The method is applied to the case of a symmetrical Joukowski profile whose aileron is vibrating harmonically. The calculation of the pressure on the surface is carried out, to the first order, under the assumption that both the aileron angle and the airfoil thickness are small. *W. R. Sears* (Ithaca, N. Y.).

Robinson, A. Wave reflection near a wall. Coll. Aéronaut. Cranfield. Rep. no. 37, 26 pp. (2 plates) (1950).

The author considers the wave reflection from the boundary layer near a wall by considering inviscid and irrotational disturbances and reaches the conclusion that the disturbance does not propagate far upstream if the velocity distribution is continuous in the boundary layer. A later paper by Lighthill [see the following review] gives a better treatment of the problem with similar conclusions. However, the linearization used in these analyses presumably prevented an adequate account to be obtained for all the main features of the observed phenomena. *C. C. Lin.*

Lighthill, M. J. Reflection at a laminar boundary layer of a weak steady disturbance to a supersonic stream, neglecting viscosity and heat conduction. Quart. J. Mech. Appl. Math. 3, 303–325 (1950).

The author develops a comprehensive linear theory for the reflection of a weak steady disturbance at a boundary layer with supersonic free stream, neglecting viscosity and heat conduction. Fourier analysis is used. For the treatment of the reflection of a discontinuity (say of pressure), the differential equation for the amplitude function is treated asymptotically by Langer's theory. The result indicates that a discontinuous rise in pressure (or other similar quantities) is reflected as a ridge in pressure. The effect of upstream propagation is discussed by an investigation of the singularity of the integrand in the Fourier representation of the pressure (say). It is shown that the propagation cannot extend very far upstream within the linear theory. The reason why this result is at variance with the conclusion of Tsien and Finston is given. In trying to apply the theory to phenomena of shock-boundary layer interaction, it is noted that the linear theory will never be able to explain the observed pressure rise at the wall. Four possible mechanisms are suggested to explain this difficulty.

C. C. Lin (Cambridge, Mass.).

Sakurai, Akira. On the theory of cylindrical shock wave. J. Phys. Soc. Japan 4, 199–202 (1949).

A radial flow of a compressible fluid with constant viscosity and heat conduction coefficients is considered. The equations of motion are integrated numerically by taking special values of the mass flow and the Prandtl number. The solution admits three different types of flows, namely: (a) the velocity gradient is entirely negative; (b) it is first negative and then positive; and (c) it is positive in the whole flow field. The first type corresponds to flows that are subsonic at infinity and supersonic in the neighborhood of the origin. The second corresponds to flows that are subsonic at infinity, become supersonic and then quickly return to subsonic flow. The third type is subsonic near the origin but supersonic away from it. [Except for the first type, the physical existence of these flow types is questionable. Furthermore, the author's interpretation of the pressures in a Laval nozzle in the light of these results is difficult to follow.] *Y. H. Kuo* (Ithaca, N. Y.).

Popov, S. G. Examples of the exterior problem of the aerodynamics of a very rarified gas. Amer. Math. Soc. Translation no. 36, 20 pp. (1951).

Translated from Vestnik Moskov. Univ. 3, no. 5, 25–37 (1948); these Rev. 11, 222.

Beatty, Ralph E., Jr. Boundary layer attenuation of higher order modes in rectangular and circular tubes. J. Acoust. Soc. Amer. 22, 850–854 (1950).

The author evaluates the attenuation of acoustic waves in ducts arising from boundary layer losses by joining on to the usual nonattenuating duct solutions a highly damped wave within the boundary layer. The change in the normal acoustic admittance v_n/p, where v_n is the component of the velocity normal to the duct walls, may then be computed and consequently the attenuation. The calculations are carried out for tubes of rectangular and circular cross-section. *H. Feshbach* (Cambridge, Mass.).

Vaĭnštein, L. A. The theory of sound vibration in open tubes. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 19, 911–930 (1949). (Russian)

Il potenziale di velocità, rispettivamente all'interno e all'esterno del tubo, viene scritto nella forma

$$\Phi(r, z) = -i2\pi^2 a \int_C e^{i\omega z} v \begin{Bmatrix} J_0(vr)H_1^{(1)}(va) \\ J_1(va)H_0^{(1)}(vr) \end{Bmatrix} F(w)\, dw$$

essendo r, z coordinate cilindriche, a il raggio del tubo, $v=(k^2-w^2)^{\frac{1}{2}}$ e C un opportuno cammino d'integrazione. Se il tubo occupa l'intervallo $0<z<\infty$, si hanno per $r=a$ le due condizioni $f(z)=\int_C e^{i\omega z}F(w)\,dw=0$ per $z<0$, $\int_C e^{i\omega z}L(w)F(w)\,dw=0$ per $z>0$ con $L(w)=\pi v^2 a J_1(va)H_1(va)$. Inoltre, ammettendo la presenza di dissipazione, $f(z)$ dovrà risultare della forma $f(z)=Ae^{-ihz}+\hat{f}(z)$ dove h rappresenta il numero d'onde del modo incidente all'interno del tubo e $\hat{f}(z)$ una funzione che si annulla per $z=\pm\infty$. Con pure considerazioni di teoria delle funzioni l'autore riesce allora a determinare il cammino C (costituito dall'asse reale e da un cappio che abbraccia il punto $w=-h$) e la struttura della funzione incognita $F(w)$. I risultati vengono utilizzati

per il calcolo dei coefficienti di riflessione dei vari modi nel tubo, della caratteristica d'irraggiamento, delle curve di risonanza dei risonatori cilindrici aperti a un estremo.

G. Toraldo di Francia (Firenze).

Osadčenko, A. F. Diffraction of acoustic waves in tubes of variable diameter. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 616–633 (1949). (Russian)

I tubi considerati hanno simmetria di rivoluzione attorno all'asse x. La sezione meridiana della parete ha l'equazione $r_0 = R_0 e^{p(x)}$, essendo r_0 la distanza dall'asse e R_0 una costante. Nel piano meridiano si prendono come linee coordinate le rette x = costante e le linee $r = Re^{p(x)}$. L'autore scrive l'equazione differenziale a cui soddisfa la pressione acustica p_a in queste coordinate curvilinee. Quando $p(x)$ è ovunque molto piccola (tubo quasi cilindrico), l'equazione si semplifica in

$$\frac{\partial^2 p_a}{\partial t^2} e^{2p(x)} = c^2 \left\{ \frac{\partial}{\partial x}\left(e^{2p(x)} \frac{\partial p_a}{\partial x} \right) + \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial p_a}{\partial R} \right) \right\}$$

e viene applicata al caso della tromba esponenziale e della tromba conica. L'autore mette in rilievo l'esistenza di superficie di discontinuità della fase, lungo le quali, nei casi reali, nasce un moto di rotazione e non è più possibile valersi del potenziale di velocità. *G. Toraldo di Francia* (Firenze).

Mendousse, J. S. On the theory of acoustic radiation pressure. Proc. Amer. Acad. Arts Sci. **78**, 148–164 (1950).

The author obtains results concerning the acoustical radiation pressure $\overline{\Delta p}$ in infinitesimal irrotational oscillations of a perfect fluid obeying a barotropic relation $\rho = \rho(p)$. His method is to expand all quantities in power series in a constant parameter A, "a generalized amplitude, . . . not necessarily . . . any physical amplitude." He retains the first two terms in these expansions. His result, which is independent of A, is

$$\overline{\Delta p} = V^{-1}[c^2 \overline{\Delta M} - (3-2D)\bar{E}_p + \bar{E}_k] + \bar{e}_p - \bar{e}_k,$$

where $\bar{e}_p$ and $\bar{e}_k$ are the time averages of the potential and kinetic energy densities per unit volume, respectively, at the boundary point where $\overline{\Delta p}$ is to be calculated; $\bar{E}_p$ and $\bar{E}_k$ are the time averages of the total kinetic and potential energies in an arbitrary internal volume V; c is the velocity of sound in the undisturbed fluid; D (apparently) is defined as $1 - \frac{1}{2}\rho^2 c^2 d^2\rho/dp^2$ evaluated in the undisturbed fluid; and $\overline{\Delta M}$ is the time average mass increase in V. The author gives an approximate formula for $\overline{\Delta M}$ which the reviewer cannot understand. The author states that in general all quantities except $\overline{\Delta M}$ which occur in his result may be calculated from a solution of the ordinary wave equation appropriate to the boundary in question, but that a higher order approximation must be used to calculate $\overline{\Delta M}$. He considers some simple special cases in which it is not necessary to solve any boundary problem. The author discourages the reader by using undefined terms and changing his notations without warning. *C. Truesdell* (Bloomington, Ind.).

Rłevkin, S. N. On the flux of energy in the field of a spherical sound radiator. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 1380–1396 (1949). (Russian)

Espresso il potenziale di velocità mediante la nota serie (di prodotti di funzioni sferiche per funzioni di Bessel e di Neuman di ordine $m+\frac{1}{2}$), vengono calcolate le componenti radiale e tangenziale del flusso di energia. Il flusso tangenziale non ha luogo per i singoli modi elementari (corrispondenti a un dato valore di m), mentre appare per interferenza quando si sovrappongono più modi. L'autore calcola l'energia cinetica immagazzinata dalla componente reattiva della velocità sonora. *G. Toraldo di Francia.*

Anderson, Victor C. Sound scattering from a fluid sphere. J. Acoust. Soc. Amer. **22**, 426–431 (1950).

The author considers the scattering of sound from a fluid sphere whose acoustic properties are near that of the medium in which they are imbedded. Sphere diameters up to several wavelengths are discussed. The excess sound pressure satisfies the wave equation: $\nabla^2 p = (1/c_1^2)\partial^2 p/\partial t^2$, where c_1 is the sound velocity within the sphere and c_2 is the velocity in the medium external to the sphere. The pressure as well as the radial velocity must be continuous at surface of the sphere while at large distances from the sphere the pressure should consist of an incident plane wave plus a diverging scattered wave. The solution is determined by expanding the solution inside and outside the sphere in terms of Legendre polynomials and the appropriate spherical Bessel functions; the unknown coefficients being determined by the boundary conditions. The cross-section may then be evaluated, but except for limiting cases this is done numerically.

H. Feshbach (Cambridge, Mass.).

Polubarinova-Kočina, P. Ya. On the unsteady motion of ground water. Doklady Akad. Nauk SSSR (N.S.) **75**, 357–360 (1950). (Russian)

Formulae are derived on the basis of which a graphic method can be developed for the determination of the non-stationary motion of ground water near a canal that is being filled with water. *H. P. Thielman* (Ames, Iowa).

Polubarinova-Kočina, P. Ya. On the influence of the slope of the water bed and of infiltration on unsteady motion of ground water. Doklady Akad. Nauk SSSR (N.S.) **75**, 511–514 (1950). (Russian)

The problem of determining the height of the ground water is reduced to the solution of a linear partial differential equation with certain boundary conditions. Explicit solutions are given for various conditions which fall under the case mentioned in the title. *H. P. Thielman.*

Hidaka, Koji. Lateral mixing and wind-driven currents in an enclosed ocean. J. Marine Research **9**, 55–64 (1950).

The vorticity equation for the mass transport in a rectangular ocean is set up including lateral diffusion and surface stress. A surface stress is assumed as it might be produced by the anticyclonic wind systems, and the equation is integrated with the boundary conditions that the derivatives of the stream function vanish at the boundaries of the ocean. An approximate solution is obtained by Galerkin's method. The result indicates two large current systems in an ocean of the approximate dimensions of the North Pacific, a clockwise gyral in the western half and a counter-clockwise gyral in the eastern half. The western gyral becomes relatively more intense with increasing lateral mixing. *H. Panofsky* (New York, N. Y.).

Godson, Warren L. Generalized criteria for dynamic instability. J. Meteorol. **7**, 268–278 (1950).

The motion of an air-parcel which is given an impulse (perturbed parcel) is compared with the motion of an air-parcel of the environment which is unperturbed. The equations of motion of the perturbed parcel are expressed

in terms of relative rectangular coordinates (x', y', z') with origin at the unperturbed parcel. The equations are linearised and solutions are sought for in which the relative coordinates depend on the time t like $\exp(\pm i\omega t)$. It is shown that, in general, $i\omega$ is a complex number. The author then assumes that, if the real part of $i\omega$ is not zero, the motion of the perturbed parcel is unstable, i.e., (x', y', z') must increase indefinitely with t. The reviewer does not understand this assumption: if the real part of $i\omega$ is positive, instability occurs; but if it is negative, the motion is stable (e.g., damped oscillations take place). For example, from the author's formulae (36) and (37), it follows that x' and y' tend to zero as t tends to infinity provided that the arbitrary constants are chosen so that $A = -B_1 = B_2$ and $C = -D_1 = D_2$. It is not self-evident, nor is it proved, that these constants cannot be related in this way. Thus (36) and (37) apparently include stable motions, contrary to the author's conclusion. The criteria of instability he obtains depend on the author's assumption with regard to the real part of $i\omega$, and involve the 3-dimensional gradients of temperature and geostrophic wind. *G. C. McVittie* (London).

Elasticity, Plasticity

***Sokolnikoff, I. S. Approximate methods of solution of two-dimensional problems in anisotropic elasticity.** Proc. Symposia Appl. Math. **3**, 1–11. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

This paper summarises part of a research made to develop practical methods of solving plane stress and plane strain problems in anisotropic elasticity, and also problems concerning the deflections of thin anisotropic plates.

A brief outline is given of the method proposed by S. G. Lechnitzky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] N.S. **1**, 77–90 (1937), p. 81; **2**, 181–210 (1938)] to solve two-dimensional boundary-value problems in anisotropic elasticity associated with the differential equation

$$c_1\frac{\partial^4 U}{\partial x^4}+c_2\frac{\partial^4 U}{\partial x^3\partial y}+c_3\frac{\partial^4 U}{\partial x^2\partial y^2}+c_4\frac{\partial^4 U}{\partial x\partial y^3}+c_5\frac{\partial^4 U}{\partial y^4}=0,$$

where $U(x, y)$ is Airy's stress function. This method extends that of Muschelišvili [Math. Ann. **107**, 282–312 (1932)] for isotropic elasticity. Owing to computational difficulties involved in the solution of actual problems, the author presents perturbation methods of solution. These are illustrated with reference to the differential equation

$$c_1\frac{\partial^4 U}{\partial x^4}+c_3\frac{\partial^4 U}{\partial x^2\partial y^2}+c_5\frac{\partial^4 U}{\partial y^4}=0$$

for two-dimensional problems in orthotropic elasticity. The perturbation methods are apparently quite successful and the author states that they give useful results even for highly anisotropic materials such as wood.

H. D. Conway (Ithaca, N. Y.).

***Holl, D. L. Dynamic loads on thin plates on elastic foundations.** Proc. Symposia Appl. Math. **3**, 107–116. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

Formal solutions are given for a number of problems concerning time-dependent deflections of thin plates on an elastic foundation. The problems are simplified by the omission of inertia effects of the foundation.

E. Reissner (Cambridge, Mass.).

Chandra Das, Sisir. Note on the bending of certain thin elastic plates by concentrated loads. Bull. Calcutta Math. Soc. **42**, 89–93 (1950).

Closed-form solutions are obtained for the transverse deflection of thin plates with clamped edge when the boundary of the plate is (1) the inverse of an ellipse, (2) an elliptic limaçon, and a concentrated load is acting at the origin and the focus of the limaçon, respectively. For the method of solution reference is made to B. Sen [Indian Phys.-Math. J. **5**, 17–20 (1934)]. *E. Reissner* (Cambridge, Mass.).

***Perry, C. L. The bending of thin elliptic plates.** Proc. Symposia Appl. Math. **3**, 131–139. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

Equations for the bending of plates are presented in elliptic coordinates. A Mathieu function method of solving these equations for elliptic plates under lateral loads is described, where the plate deflection is small compared with the plate thickness. The lateral pressure is expressed as a series of Mathieu functions in this solution. An equivalent solution in a series of hyperbolic and trigonometric functions is also described. The extension of these methods to the case where the plate deflection is of the order of the plate thickness is indicated. *S. Levy* (Washington, D. C.).

Mitra, D. N. Corrections to my paper "On the flexure problem of a limacon and some other boundaries." Bull. Calcutta Math. Soc. **42**, 188 (1950).

Cf. the same Bull. **41**, 153–158 (1949); these Rev. **11**, 287.

Sugamoto, Shinsuke. Theory of elasticity for finite displacement. J. Phys. Soc. Japan **3**, 90 (1948).

Sugamoto, Sinsuke. Theory of elasticity for finite displacement. II. J. Phys. Soc. Japan **4**, 348–349 (1949).

The author writes down four forms of the differential equations and boundary conditions (for applied surface forces) in the classical finite strain theory of elasticity, in the case when both the Eulerian and the Lagrangean coordinates are rectangular Cartesian. His equations (L) and (E) employ, respectively, the strain tensors defined by $ds^2-dS^2=2E_{ij}dX^i dX^j=2e_{ij}dx^i dx^j$, where capital letters refer to Eulerian, small letters to Lagrangean variables. His equations (L') and (E') employ, respectively, the tensors reciprocal to E_{ij} and to e_{ij}. The stress-strain relations corresponding to equations (L) were discovered by Kelvin [Papers, vol. 3, 351–394, 1890, see § 62], Boussinesq [C. R. Acad. Sci. Paris **71**, 400–402 (1870)], and E. and F. Cosserat [Ann. Fac. Sci. Toulouse (1) **10**, no. I (1896), see § 24]; those corresponding to (L') and (E), by Murnaghan [Amer. J. Math. **59**, 235–260 (1937)]; those corresponding to (E'), by Finger [Akad. Wiss. Wien. S.-B. IIa **103**, 1073–1100 (1894)]. The equations (E) and (E') are valid only for isotropic media and should be reduced to the more explicit form given by Finger [loc. cit.]. *C. Truesdell.*

Swainger, K. H. Finite elastic straining. Appl. Sci. Research A. **2**, 281–298 (1950).

The author claims that finite elastic strain can be described by the equations of the classical infinitesimal theory, provided stress and coordinates be referred to the deformed body and provided there exist one point at which the rotation vanishes. He gets excellent agreement between the results of experiment and of his theory in the case of a stretched rubber sheet containing a circular hole. [To the reviewer it seems that what the author offers as a derivation

(§ 4) amounts to establishing his equations by fiat, and that the equations themselves are wholly false.]

C. Truesdell (Bloomington, Ind.).

†**Seth, B. R. Some recent applications of the theory of finite elastic deformation.** Proc. Symposia Appl. Math. 3, 67–84. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

This paper purports to give a summary of the successes of "the" theory of finite strain when applied to special problems. By "the" theory the author apparently means that based upon the stress-strain relations proposed by himself [Philos. Trans. Roy. Soc. London. Ser. A. **234**, 231–264 (1935); and numerous later papers], but he makes no attempt to refute the serious (though only partially just) objection raised against this theory by Murnaghan [Amer. J. Math. **59**, 235–260 (1937), see § 3], Zvolinsky and Riz [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] N.S. **2**, 417–426 (1939)], Signorini [Atti Secondo Congresso Un. Mat. Ital. Bologna. 1940, pp. 56–71, Cremonense, Rome, 1942, see p. 69; these Rev. **8**, 421], and Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. **241**, 379–397 (1948), see § 5; these Rev. **10**, 340]. His summary of the work of others is neither complete nor accurate. After a dubious argument concerning the value to be assigned to Young's modulus, based upon his own formula for simple extension, he states that the formula of James and Guth [Physical Rev. (2) **59**, 111 (1941)] (which he attributes to Treloar) gives results "very much like" his own, while a glance at the two formulae shows that the one predicts a tension of one half the value of Young's modulus for infinite stretch, while the other predicts infinite tension for infinite stretch. The author's summary of Murnaghan's results on hydrostatic pressure does not mention his latest and best work [Proc. Nat. Acad. Sci. U. S. A. **30**, 244–247 (1944); these Rev. **6**, 195], which was presented by Murnaghan at the same symposium two years earlier [Proc. Symposia Appl. Math., vol. I, pp. 158–174, Amer. Math. Soc., New York, 1949; these Rev. **11**, 67]. While the author's criticism of one of the proposals of Zvolinsky and Riz is well taken, he does not mention that the views of these authors, expressed in a long series of notes, are continually changing. In his discussion of special problems the author does not mention the serious and careful work of Signorini [op. cit. and later papers] on simple extension, flexure of a rectangular bar, and thermoelastic effects. Most shocking is the author's failure to give any account whatever of the general exact solutions, valid for any type of strain energy function, for numerous problems of large strain of incompressible materials which have been obtained by Rivlin [op. cit. and later papers]. The bibliography contains 49 items, but to most of them, including some papers of Signorini and that of Rivlin already mentioned, there is a mere citation, without any attempt to summarize their content.

The topics treated are: components of strain; consistency equations; stress-strain relations; simple tension; hydrostatic pressure; pure flexure; torsion and flexure; shells and tubes under uniform pressure; elastic failure; vibrational problems; nonisotropic bodies; miscellaneous problems.

C. Truesdell (Bloomington, Ind.).

Berti, Giuliana. Il problema di Saint-Venant nell'elasticità ereditaria. Boll. Un. Mat. Ital. (3) **5**, 139–144 (1950).

The author applies Volterra's accumulative theory of elasticity to the deformation to straight bars loaded at their ends. Closely guided by St. Venant's results in the classical theory, she obtains solutions for extension, bending by a terminal couple parallel to a principal axis of inertia of the section, and torsion. The displacements are of St. Venant's type except for a multiplicative function of time. The results are presented in terms of operators defined by means of Volterra's coefficients of accumulation and of the inverses of these operators (not shown to exist). *C. Truesdell.*

Kleĭn, G. K. On the equations proposed by Dr. Ing. O. K. Fröhlich. Vestnik Inženerov i Tehnikov **1948**, no. 2, 71–74 (1948). (Russian)

It is pointed out that the formulas proposed by Fröhlich for the stresses in a half-space fail, in general, to satisfy the compatibility conditions. However, if Young's modulus is assumed in the form $E=E_0Z^{\nu-3}$, where Z is the depth from the surface, a relationship exists between ν and the Poisson ratio μ for which all of the elastic equations are satisfied. In particular, it is shown that for a three-dimensional stress distribution the compatibility equations are satisfied for all μ if $\nu=2$ and for other values of ν if $\mu=(\nu-1)^{-1}$. For the plane strain case the corresponding conditions are $\nu=3$ and $\mu=(\nu-1)^{-1}$ and for plane stress $\nu=3$ and $\mu=(\nu-2)^{-1}$.

H. I. Ansoff (Santa Monica, Calif.).

Freiberger, W. On the solution of the equilibrium equations of elasticity in general curvilinear coordinates. Australian J. Sci. Research. Ser. A. **2**, 483–492 (1949).

The author shows that a necessary and sufficient condition for a vector field u_r to be a solution of Navier's statical equations in the infinitesimal theory of elasticity, extraneous forces being zero, is

$$\mu u_r=\phi_r=F_{,r}, \quad F=\phi_0+\frac{\lambda+\mu}{\lambda+2\mu}q^r\phi_r,$$

where ϕ_0 is a harmonic scalar, ϕ_r is a harmonic vector, and q^r is the contravariant radius vector. He shows that it is always possible to set one of the three functions ϕ_r equal to zero. He gives expressions for the stress tensor and, in the case of orthogonal coordinates, for the physical components of stress. He shows that his result is essentially a generalization to curvilinear coordinates of that of Papkovitch [C. R. Acad. Sci. Paris **195**, 513–515, 754–756 (1932)] and Neuber [Z. Angew. Math. Mech. **14**, 203–212 (1934)], and that his harmonic field ϕ_r is essentially the Laplacian of the biharmonic Galerkin vector [C. R. Acad. Sci. Paris **190**, 1047–1048 (1930)].

C. Truesdell (Bloomington, Ind.).

Goss, R. N. Center of flexure of a triangular beam. Proc. Amer. Math. Soc. **1**, 744–750 (1950).

The flexure solution of the reviewer [Proc. London Math. Soc. (2) **41**, 323–331 (1936)] is used to determine the center of flexure of any isosceles triangular section for any value of the Poisson's ratio σ. Numerical results are obtained for y_0, the distance of the center of flexure from the vertex of the triangle and graphs are drawn to exhibit the variation of y_0 with σ for selected values of the semi-vertical angle β. It is found that unless β is small y_0 varies very little with σ, thus giving a theoretical justification for the practice of choosing arbitrary values of σ in the solution of engineering problems. It is also found that the results obtained agree very well with known experimental results.

B. R. Seth (Delhi).

***Hay, G. E. Beams under concentrated loading.** Proc. Symposia Appl. Math. 3, 13–26. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

Mushelišvili's method is used to discuss the plane strain problem of a long cylinder bounded by confocal ellipses and acted upon by two equal concentrated forces at the ends of the major axis. An infinite set of linear equations is obtained, and the difficulties in obtaining a convergent solution are found to be of the same type as those encountered when the method of curvilinear coordinates is used. Some calculations are carried out by using a measure of the eccentricity of the bounding ellipses as a perturbation parameter.

B. R. Seth (Delhi).

Gorbunov-Posadov, M. I. Calculation of an infinite beam resting on the elastic semi-space. Engineering Rev. [Akad. Nauk SSSR. Inženernyĭ Sbornik] 3, no. 2, 85–99 (1947). (Russian. English summary)

Consider an infinite elastic beam under the action of a concentrated force resting on an elastic homogeneous half-space. Assuming the beam to be rigid in the lateral direction, the author uses Boussinesq's formula to obtain approximate solutions for the displacement, sheer force, and moment distribution in the beam. The details of the computation and auxiliary tables are presented for one approximate solution. The degree of approximation is not estimated.

H. I. Ansoff (Santa Monica, Calif.).

Gorbunov-Posadov, M. I. Semi-infinite beam on the elastic semi-space. Akad. Nauk SSSR. Inženernyĭ Sbornik 4, no. 1, 115–136 (1947). (Russian. English summary)

The article outlines the calculation of a semi-infinite beam on an elastic semi-space, the bar being loaded with a concentrated force or with a force uniformly distributed along the whole beam. Parts 1 and 2 of the article give a theoretical solution of the problem. Part three gives formulae and tables for practical computations. *Author's summary.*

Joseph, J. A., and Brock, J. S. The stresses around a small opening in a beam subjected to pure bending. J. Appl. Mech. 17, 353–358 (1950).

Greenspan [Quart. Appl. Math. 2, 60–71 (1944); these Rev. 5, 250] solved the problem of an infinite plate under generalized plane stress, subjected to uniform stress at infinity and having a hole in it with free boundaries, the boundary of the hole being a special ovaloid. Morkovin [ibid., 350–352 (1944); these Rev. 6, 196] then solved the same problem more concisely by use of the method of Mushelišvili. In the present paper the authors solve the problem which differs from the above only in that the stresses at infinity are those found in the web of a beam under pure bending. The method used is that employed by Morkovin. Numerical examples are included.

G. E. Hay (Ann Arbor, Mich.).

***Poritsky, H. Stress fields of axially symmetric shafts in torsion and related fields.** Proc. Symposia Appl. Math. 3, 163–186. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

In the torsion of axially symmetric shafts of variable cross section two functions $\phi(r, z)$ and $\psi(r, z)$ in cylindrical coordinates represent the angular displacement and the stress function, from which the stresses are $r^2T_{z\theta}=\mu\psi_r=\mu r^3\phi_z$, $r^2T_{r\theta}=-\mu\psi_z=\mu r^3\phi_r$; subscripts on ψ and ϕ denote partial derivatives and μ is the shear modulus. The function ϕ satisfies $(r^3\phi_r)_r+(r^3\phi_z)_z=0$ and the boundary conditions agree with external forces applied on the ends of the shaft or on the lateral surfaces. The solutions ϕ are identifiable with harmonic functions in a five-dimensional space. The author develops these harmonic solutions ϕ and the "conjugate" ψ by methods analogous to those used for obtaining harmonics in two and three dimensions by integral representations of analytic functions, spherical harmonics, and Legendre polynomials. Suitable combinations of these five-dimensional harmonics solve the torsion of axially symmetric shafts which approximate practical cases. Bers and Gelbart [Quart. Appl. Math. 1, 168–188 (1943); these Rev. 5, 25, 328] and Weinstein [ibid. 5, 429–444 (1948); Studies and Essays Presented to R. Courant . . . , pp. 451–460, Interscience, New York, 1948; these Rev. 10, 116] solve similar torsion problems, and the former give a treatment of ϕ and ψ in terms of "sigma monogenic" functions analogous to analytic functions of a complex variable.

D. L. Holl (Ames, Iowa).

Goodier, J. N. Elastic torsion in the presence of initial axial stress. J. Appl. Mech. 17, 383–387 (1950).

This paper contains an extension of the Saint-Venant solution for ordinary torsion to the problem of torsion in the presence of any distribution of initial axial stress over the cross-section. The equilibrium equations and boundary conditions of Trefftz [Verh. 3. Int. Congr. Tech. Mech. III, 44–50 (1931); Z. Angew. Math. Mech. 13, 160–165 (1933)] and Kappus [ibid. 19, 271–285 (1939); these Rev. 1, 92; see also Biot, J. Appl. Phys. 10, 860–864 (1939); these Rev. 1, 91; Neuber, Z. Angew. Math. Mech. 23, 321–330 (1943); these Rev. 7, 40; Prager, Quart. Appl. Math. 4, 378–384 (1947); these Rev. 9, 120; and the author, Cornell University Engineering Experiment Station, Bulletin no. 27, Ithaca, N. Y., 1941; ibid., no. 28, Ithaca, N. Y., 1942; same J. 9, A-103–A-107 (1942); these Rev. 4, 64] become the same as for the Saint-Venant theory if the usual displacements of the latter theory are prescribed. The central difference arises in the torque required to maintain this state of torsion. The additional couple, over the pure torsional couple when axial stresses are lacking, is $T_3=\alpha\iint S_{zz}r^2\,dA$, where A = area of section, α = angle of twist, S_{zz} = axial stress, $r^2=x^2+y^2$ with the centroid as origin. This confirms that a constant axial tension increases the torsional rigidity and thrust decreases it. The author gives the details for the case of initial bending with torsion and some experimental results confirming the analysis. *D. L. Holl* (Ames, Iowa).

Ōkubo, H. Torsion of a circular shaft with a number of longitudinal notches. J. Appl. Mech. 17, 359–362 (1950).

The torsion of a circular shaft with m equally spaced boundary notches which are approximately semicircular is solved by means of elementary harmonic functions which conveniently simplify the numerical calculations. In case $m=1$ the solution agrees with Weber's case of a semicircular notch centered on the boundary of the shaft. When $m\geqq 2$ the notches are not strictly in the form of a semicircle and this accounts for the simplification in the calculations. The author finds the values of the maximum shearing stresses which occur at the bottom of the groove in the cases $m=2, 3, 4$. For fixed value of shearing modulus and angle of twist, the corresponding torsion moments are computed. An extension of the method is made to circular shafts with shallow notches by satisfying the boundary condition along the notch at a finite number of points. An application

is made to a circular shaft which has two equal symmetrical segments removed from the section. *D. L. Holl.*

Mindlin, R. D., and Goodman, L. E. Beam vibrations with time-dependent boundary conditions. J. Appl. Mech. 17, 377–380 (1950).

The authors consider a prismatic beam executing flexural vibrations according to the simple Bernoulli-Euler theory, in the case when the boundary conditions involve known functions $f_i(t)$, $i=1, 2, 3, 4$, where t is the time. The deflection of the beam is $w(x, t)$. The following transformation is introduced: $w=\zeta(x, t)+f_1g_1+f_2g_2+f_3g_3+f_4g_4$, where g_i are unspecified functions of x. By proper choice of $g_i(x)$ the problem for the determination of $\zeta(x, t)$ is made the same as that above for the determination of $w(x, t)$, except that t no longer appears in the boundary conditions. The function $\zeta(x, t)$ is then determined by the separation of variables. The method can apparently be applied to other vibration problems. *G. E. Hay* (Ann Arbor, Mich.).

Newlands, Margery. Rayleigh waves in a two-layer heterogeneous medium. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. 6, 109–124 (1950).

L'auteur étudie la propagation des ondes de Rayleigh (ondes élastiques superficielles) dans un milieu homogène infini limité par un plan et recouvert d'une couche de densité constante dans laquelle la rigidité varie linéairement avec la profondeur. Si cette couche est incompressible, on doit résoudre une équation différentielle du second ordre dont la solution a été exprimée par Stoneley au moyen des fonctions $W_{k,\frac{1}{2}}$ de Whittaker (fonctions hypergéométriques confluentes). L'auteur la développe au contraire en série de puissances du nombre d'ondes. Pour une application aux ondes longues des séismes il suffira de prendre quatre ou cinq termes. Le cas où la compressibilité croît proportionnellement à la rigidité conduit à deux équations simultanées du second ordre qui peuvent être résolues de façon analogue. Dans l'un ou l'autre cas, l'accord avec les observations de Röhrbach est très médiocre. La théorie pourrait aussi être utilisée pour exprimer l'effet, sur les microséismes à courte période, d'une couche superficielle se consolidant progressivement en profondeur. *J. Coulomb* (Paris).

Nekrasov, A. I. Application of the theory of integral equations to the calculation of the critical velocity of flutter of an airplane wing. Akad. Nauk SSSR. Inženernyĭ Sbornik 4, no. 1, 3–45 (1947). (Russian. English summary)

In the present work the problem of flutter of a wing is reduced to the solution of a system of two homogeneous linear integral equations. The distinguishing feature of the method based on the integral equations is that preliminary determination of the forms of bending and twisting of the airplane wing are unnecessary. The work contains a detailed description of the procedure to be followed in obtaining numerical results. *From the author's summary.*

Hodge, P. G., Jr. An introduction to the mathematical theory of perfectly plastic solids. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Rep. A 11-S2, iii+396 pp. (1950).

A very clear and readable introduction to the theory of plasticity and in particular to the known basic solutions of problems of torsion, plane stress, and plane strain. Numerous references are listed at the end of each of the eleven chapters. *E. Reissner* (Cambridge, Mass.).

Gubanov, A. I. The mechanics of elastic-viscous-plastic bodies. I. Generalized equations of motion. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 19, 34–42 (1949). (Russian)

The behavior of homogeneous isotropic deformable bodies is discussed in terms of lumped-parameter mechanical models. The models are composed of series and parallel combinations of springs, dashpots, and "dry friction" elements which are introduced to account for the yield point in plastic behavior. Linear relations between the stress and strain deviators are assumed for all of these elementary deformation mechanisms. It is shown that under this assumption the most general model leads to a linear relation of the type (1) $\Delta_1 S=S_0+\Delta_2 E$, where $\Delta_i=\alpha_i+\tau_i d/dt+\nu_i d^2/dt^2+\cdots$, S and E are the stress and strain deviators, respectively, and S_0 is the value of S at the yield point. For purposes of the present discussion only terms through the second order are retained in Δ_i. It is shown that, by assigning appropriate values to α_i, τ_i, ν_i ($i=1, 2$), and S, equation (1) can be made to describe six useful types of behavior in deformation: (a) Maxwell's liquid (viscosity η and stiffness G in series); (b) Kelvin's body (viscosity and stiffness in parallel); (c) Frenkel's body (a series pair of η and G in parallel with a G); (d) Frenkel's liquid (the previous combination in series with an η); (e) plastic body with hardening (no viscosity); (f) Bingham body (a parallel pair of η and dry friction element in series with a G).

In discussion of models exhibiting plastic behavior with hardening, no provision is made to distinguish between loading and unloading conditions. Equation (1) is next used to derive a modified Navier-Stokes equation which introduces time operators on the divergence and the Laplacian terms. The Hencky-Mises yield condition is assumed for the determination of S_0. *H. I. Ansoff.*

Gubanov, A. I. The mechanics of elastic-viscous-plastic bodies. II. Shear. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 19, 43–61 (1949). (Russian)

A generalized linear equation relating the deviators of stress and strain in viscous-plastic-elastic bodies was obtained by the author [see the preceding review]. The present paper presents an application of this equation to a study of pure shear of a layer of thickness h. The boundary values are: (a) displacement zero at $z=0$; and (b) either displacement or stress prescribed at $z=h$. Four types of boundary values at $z=h$ are considered. The first three are: $p=$a constant; $p=At$; $p=A\sin\omega t$; where p is the stress (or the displacement) and A and ω are constants. The fourth is produced by a sudden adhesion of the top surface $z=h$ to a mass m per unit area moving initially with a velocity v_0. The equations are solved by means of Laplace transforms and the resulting expressions are reduced to descriptions of five distinct modes of behavior of deformable bodies described in the paper reviewed above. The reader is advised to take the solution for the case of a plastic body with reservations, since it appears to this reviewer that the unloading conditions are improperly accounted for [cf., for example, the expression following equation 57]. *H. I. Ansoff* (Santa Monica, Calif.).

Gubanov, A. I. The mechanics of elastic-viscous-plastic bodies. III. Torsion of a circular cylinder. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 19, 773–781 (1949). (Russian)

The present paper [see the two preceding reviews] considers torsion of a circular cylinder with prescribed time-dependent angle of torsion θ. Denote the differential operator

$\alpha_1+\beta_1\partial/\partial t+\gamma\partial^2/\partial t^2$ by Δ_1; then the stress-strain relation is given by $\Delta_1\sigma=S_0+\Delta_2\epsilon$, where S_0 is constant and σ and ϵ are the stress and the strain, respectively. Laplace transforms are used to obtain general formulas for the resulting stress distribution in the plastic and the elastic zone for the following cases: $\theta=$constant; $\theta=At$; $\theta=A\sin\omega t$. By assigning special values to the constants characterizing the body, the author reduces the formulas to the six cases described in the second preceding review. *H. I. Ansoff.*

Gubanov, A. I. The mechanics of elastic-viscous-plastic bodies. IV. Stretching and compression of cylinders. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 892–910 (1949). (Russian)

This is the fourth and last in a series of papers dealing with what the author calls "equations of motion of a generalized elastic-viscous-plastic body" [see the three preceding reviews]. The present paper deals with a uniaxial uniform state of stress in a cylindrical body. The boundary values, the method of solution, and the final results are similar to those obtained in the previous papers. *H. I. Ansoff.*

Castiglia, Cesare. Il fenomeno del ritiro nei solidi eterogenei e la conseguente variazione del regime statico. Atti Accad. Sci. Lett. Arti Palermo. Parte I. (4) **8**, 71–81 (1949).

A further study of the behavior of solids of a type discussed in a previous paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 311–314 (1949); these Rev. **11**, 283]. *J. L. Synge* (College Park, Md.).

Cicala, Placido. Sul carico critico di una piastra compressa oltre il limite elastico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 67–71 (1950).

The results of an earlier paper [same Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 583–586 (1950); these Rev. **12**, 373] are applied to the plastic buckling of a rectangular plate under edge thrusts. The author's main conclusion is that the buckling stress of a long-hinged flange will be as predicted by the deformation theory if the Shanley effect is taken into account. The same result was obtained by Batdorf [J. Aeronaut. Sci. **16**, 405–408 (1949)]. *W. Prager* (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

***Slyusarev, G. G. Geometričeskaya optika. [Geometrical Optics].** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1946. 332 pp. (2 plates).

The author begins with the wave equation and deduces therefrom the usual equations of geometric optics. Among the topics listed in the chapter headings are basic theories; pencils and wave surfaces; ideal and real optical systems; lenses, prisms, and plane mirrors; optical systems with spherical surfaces; bounded pencils and optical systems; aberration; the eye; properties of the image; methods of computation; imperfect optical systems; asymmetric systems; aspherical surfaces with axis of symmetry.

A. S. Householder (Oak Ridge, Tenn.).

***Hopkins, H. H. Wave Theory of Aberrations.** Oxford, at the Clarendon Press, 1950. viii+169 pp. $3.00.

The content of this book deals chiefly with geometrical optics, though the author develops in the first two chapters an approximate diffraction theory of optical instruments. His definition of aberration as given by the deviation of a wave from a spherical wave is not as original as the author assumes [cf. K. Strehl, Central-Zeitung für Optik und Mechanik, Elektrotechnik und verwandte Berufszweige **41**, 409–410 (1920)].

The first three chapters treat the optical image formation in a system with rotation symmetry by the use of the angle characteristic. The third chapter discusses the image of a small line element in and off axis (sine condition and Herschel condition). The ordering of the aberration types for finite aperture and fields into three groups in the fourth chapter, spherical aberrations, chromatic aberrations, and astigmatic aberrations, though theoretically possible, seems to the reviewer a little too formalistic.

The following chapters discuss astigmatism along a ray, first order aberration, and the Seidel theory of third order aberrations. The book shares with others the fact that the author does not seriously discuss the aberrations of skew rays. The final chapters apply the theory developed to a few classical problems. The chapters forming the end of the book, on aspheric and reflecting surfaces, could be more elaborate. *M. Herzberger* (Rochester, N. Y.).

Di Jorio, M. La teoria generale dell'isoplanatismo per aperture e campi generici. II. Fasci extrameridiani. Ottica (N.S.) **4**, 37–52 (1950).

[For part I see Ottica (N.S.) **1**, 88–116 (1947).] The author gives a generalization of his formula for the isoplanatic image of the points of an off-axis line element. He discusses especially a case in which the caustic of a point and its neighbor point are congruent [originating from each other by rotation and translation]. [The reviewer, in spite of his respect for the ingenuity of the author, must express his doubt with respect to the validity and the meaning of the results expressed for an infinitesimal change of all the rays of a finite bundle coming from an object point.]

M. Herzberger (Rochester, N. Y.).

Blottiau, Félicien. La notion d'instrument d'optique parfait. Rev. Optique **29**, 563–579 (1950).

The concept of a perfect optical system given by the so-called Gaussian approximation [reviewer's note: which incidentally cannot be realized] is not sufficient to give what could be called a perfect optical system since we have to consider the perspective under which the eye sees the image formed by the optical instrument. If one looks at an image from a distance which is very different from the perspective which the camera obtained, the perspective will be false. The author discusses these well-known facts of perspective within the approximation of Gaussian optics and makes suggestions with respect to taking care of this "error" of optical systems. *M. Herzberger.*

Jensen, H. Anallaktische Fernrohr-Systeme. Optik **7**, 16–22 (1950).

The author gives a simplified formula for the distance of the anallactic point of a telescopic system from the first part of the optical system. He finds a simple connection between the focal lengths of the objective, the distance between objective and ocular, and the distance of the focal point from the focusing device, if one demands that the anallactic

point and the length of the telescope remain constant in focusing. Several special types of analactic telescopes are discussed in detail. *M. Herzberger* (Rochester, N. Y.).

Blaisse, B. S. L'admittance optique des couches homogènes et hétérogènes. J. Phys. Radium (8) **11**, 315–320 (1950).

The author discusses the optical properties of thin films by introducing into optics the concept of impedance known in electricity. The ratio E_t/H_t, where E_t and H_t are the components of the electric magnetic vectors parallel to the plane separating two media, remains continuous even if the refractive index varies discontinuously. The author calculates the coefficient of reflection from the impedance or its reciprocal value which is called admittance, and obtains differential equations giving the change of the electric magnetic vectors as a function of the layer thickness. A special case discussed in detail is the one in which the inverse of the refractive index varies proportionally to the film thickness. *M. Herzberger* (Rochester, N. Y.).

Franz, W. On the theory of diffraction. Proc. Phys. Soc. Sect. A. **63**, 925–939 (1950).

The Kirchhoff theory of diffraction is often regarded as the first approximation in an iterative solution, but so far no one has found even a second approximation, much less proved the convergence of the process. The author shows how a consistent iterative solution can be carried out, both for sound waves and electromagnetic waves. For the sake of brevity, we consider only the acoustical case.

The essential idea, in the author's words, is as follows. "Let us take as zero order approximation a primary wave u_0 satisfying the wave equation for empty space, ignoring all diffracting objects. The next step is to take account of these neglected objects, and this may be done in a purely experimental way by some plausible modification of u_0. . . . The resulting modified function u_0' now, of course, contradicts the wave-equation in some parts of space, for instance, at the boundaries of the geometrical shadow. Thus we have next to restore the wave-equation by adding a first order correction δu_1 in order to adapt the first order function $u_1=u_0'+\delta u_1$ to the wave equation, which determines δu_1 unambiguously if only outgoing waves are taken into account. Now again, δu_1 is calculated without regard to diffracting objects; therefore u_1 has to be modified again, yielding u_1', which again violates the wave-equation and therefore demands a second order correction δu_2, and so on. The whole procedure consists . . . of two kinds of steps: 'intermediate steps' $u_r \to u_r'$ taking into account the influence of diffracting objects, and 'principal steps' $u_r' \to u_{r+1}$, adapting u to the wave-equation. The first order of this method applied to a black screen yields Kirchhoff's theory."

As the author remarks, "The intermediate steps are rather arbitrary and it requires skill to choose suitable modifications of u_r fit for the needs and conditions of a given case. We have to keep this freedom of intermediate steps in order to obtain the best possible service from our method." The method is applied to the usual test-object, the Sommerfeld half-plane. *E. T. Copson* (St. Andrews).

Toraldo di Francia, G. Sur les lois générales de la diffraction. Nuovo Cimento (9) **7**, 967–979 (1950).

A survey article treating progress since an earlier report by the author [Rev. Optique **28**, 597–611 (1949); these Rev. **11**, 293]. A bibliography of 107 items is included.

Toraldo di Francia, Giuliano. Realtà fisica di una varietà di Fermat pseudosferica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 489–494 (1950).

Fermat's principle for electromagnetic waves in an isotropic medium of variable refractive index n can be expressed in the following simple form: the light rays are geodesics in the space of the metric $ds^2=n^2(dx^2+dy^2+dz^2)$ and the electric and magnetic vectors are propagated along a ray by Levi-Civita parallel transport. Particular interest is attached to spaces of constant curvature; for example, as Luneberg pointed out [Mathematical Theory of Optics, Brown University, Providence, R. I., 1944; these Rev. **6**, 107], Maxwell's "fish eye" is simply a space of constant positive curvature. In the present note it is shown that, since the refractive index for sound waves in the sea is inversely proportional to the depth, the rays are geodesics in a space of constant negative curvature. Consequences of this result are deduced.

E. T. Copson (St. Andrews).

Caprioli, Luigi. Sulla integrazione delle equazioni di Maxwell. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **6**, 147–158 (1950).

In this paper the author discusses the direct integration of the Maxwell equations. His method is that used by Stratton [Electromagnetic Theory, McGraw-Hill, New York-London, 1941, section 8.14] and his fundamental formula is the straightforward extension to a conductive medium of a relation equivalent to Stratton's equation (19) [loc. cit.]. From this fundamental result the author derives, in the first instance for fields whose time dependence is described by a factor $e^{-i\omega t}$, and then for more general fields, the Love, Larmor-Tedone, and Tonolo formulas.

A. Erdélyi (Pasadena, Calif.).

Lošakov, L. N. On the propagation of waves along a coaxial spiral line in the presence of an electron beam. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 578–595 (1949). (Russian)

La linea è costituita da un cilindro metallico esterno e da una spirale di filo conduttore sottile, coassiale al cilindro. Lungo l'asse si propaga un sottile fascio cilindrico di elettroni. Matematicamente la spirale viene schematizzata come una parete cilindrica sottile, di conduttività perfetta nella direzione delle spire e di conduttività nulla in direzione perpendicolare ad esse. L'autore imposta direttamente le equazioni di Maxwell e l'equazione di moto degli elettroni per un sistema siffatto, nel caso della simmetria di rotazione. Mediante le condizioni al contorno del fascio elettronico, sulla spirale e sulla parete del cilindro esterno, si trova facilmente l'equazione caratteristica degli autovalori. Per procedere oltre, bisogna scendere a considerazioni approssimate. L'autore considera il caso, importante nella pratica, in cui il fascio elettronico ha densità molto piccola e la velocità degli elettroni è molto vicina alla velocità di fase delle onde libere del sistema. Sussistono allora quattro onde, di cui tre procedono in senso diretto (concorde con quello degli elettroni) e la rimanente in senso inverso. Una delle onde dirette e l'onda inversa conservano ampiezza costante lungo l'asse, mentre delle altre due la prima si va intensificando e la seconda si va indebolendo. All'entrata della linea le tre onde dirette hanno ampiezze eguali fra loro.

G. Toraldo di Francia (Firenze).

Abelès, Florin. Recherches sur la propagation des ondes électromagnétiques sinusoïdales dans les milieux stratifiés. Application aux couches minces. I. Ann. Physique (12) **5**, 596–640 (1950).

Abelès, Florin. Recherches sur la propagation des ondes électromagnétiques sinusoïdales dans les milieux stratifiés. Application aux couches minces. II. Ann. Physique (12) **5**, 706–782 (1950).

The Maxwell equations curl $\mathbf{H}=\epsilon c^{-1}\dot{\mathbf{E}}$, curl $\mathbf{E}=-\mu c^{-1}\dot{\mathbf{H}}$ are considered in a stratified medium in which ϵ and μ are functions of z. The fields considered show a simple harmonic dependence on x, y, t so that the only factors to be determined are functions of z which satisfy a system of ordinary linear differential equations with variable coefficients: (1) $U'=-ik\mu V$, $V'=-ik(\epsilon-S^2\mu^{-1})U$ is such a system (k, S are constants).

In the first paper the properties of systems of the form (1), and of the corresponding differential equations of the second order, are discussed and the results are used to investigate stratified media in which ϵ and μ are step functions, and those in which ϵ and μ vary slowly. The latter may be envisaged as a limiting case of the former. Comparisons are made with analogous systems in other branches of physics, mechanics, heat conduction, electrical transmission lines, wave mechanics, etc.

In the second paper the general theory of the first paper is applied to thin layers, of various properties and in various combinations. Both normal and oblique incidence are treated, the formulas are developed in each case, and in some cases there are also quantitative results.

A. Erdélyi (Pasadena, Calif.).

Vaĭnštein, L. A. The radiation of unsymmetrical electromagnetic waves from the open end of a circular waveguide. Doklady Akad. Nauk SSSR (N.S.) **74**, 485–488 (1950). (Russian)

L'autore estende al caso di modi elettromagnetici non simmetrici in un tubo cilindrico circolare semiinfinito lo stesso metodo applicato alla guida d'onda piana elettromagnetica [Izvestiya Akad. Nauk SSSR. Ser. Fiz. **12**, 144–165 (1948); questi Rev. **10**, 659] e alla guida d'onda acustica cilindrica [Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 911–930 (1949); questi Rev. **12**, 454], esprimendo i vettori di Hertz elettrico e magnetico per mezzo di integrali nel campo complesso e riuscendo con pure considerazioni di teoria delle funzioni a determinare il cammino d'integrazione e le funzioni incognite negli integrandi. *G. Toraldo di Francia.*

Vaĭnštein, L. A. On the diffraction of waves at the open end of a circular wave guide whose diameter is considerably greater than the wave length. Doklady Akad. Nauk SSSR (N.S.) **74**, 909–912 (1950). (Russian)

L'autore si vale delle formule esatte da lui trovate in altri lavori [vedi la precedente recensione; anche Akad. Nauk SSSR. Žurnal Tehn. Fiz. **18**, 1543–1564 (1948)] e delle espressioni asintotiche delle funzioni di Bessel per dedurre l'irraggiamento da parte di una guida d'onda cilindrica di diametro molto più grande della lunghezza d'onda.

G. Toraldo di Francia (Firenze).

Kacenelenbaum, B. Z. The symmetric excitation of an infinite dielectric cylinder. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 1168–1181 (1949). (Russian)

Il cilindro dielettrico indefinito viene eccitato da un dipolo elettrico disposto sull'asse, parallelamente ad esso. L'autore esprime il vettore di Hertz all'interno e all'esterno del cilindro mediante integrali nel campo complesso. Le funzioni integrande rimangono determinate dalle condizioni di continuità alla superficie del cilindro. Deformando il cammino d'integrazione, sì da trasportarlo all'infinito, i residui dei poli forniscono le onde piane che si propagano lungo il cilindro. Inoltre sussiste l'onda sferica che diverge dal dipolo e che può essere valutata asintoticamente mediante il metodo del colle e un'onda fortemente attenuata, che non ha effetto a distanza rilevante dal dipolo. *G. Toraldo di Francia.*

Kacenelenbaum, B. Z. The nonsymmetric excitation of an infinite dielectric cylinder. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 1182–1191 (1949). (Russian)

Questo lavoro è strettamente connesso con quello precedente. Da prima vengono discusse le radici della nota equazione caratteristica degli autovalori per il cilindro dielettrico, che corrispondono alle onde piane che si propagano lungo l'asse. Quindi l'autore, con lo stesso metodo dell'articolo precedente, studia l'eccitazione prodotta da un dipolo elettrico disposto sull'asse, perpendicolarmente ad esso. Anche in questo caso sussiste l'onda sferica, le onde piane (asimmetriche) ed un'onda fortemente attenuata.

G. Toraldo di Francia (Firenze).

Sakurai, Tokio. Theory of electromagnetic oscillation on plate conductor. J. Phys. Soc. Japan **3**, 139–143 (1948).

The author considers electromagnetic oscillations on a plate conductor whose boundary is such that the curvature is everywhere small, but is otherwise unspecified. In the neighborhood of some point P on the plate boundary a set of parabolic cylinder coordinates u_1, u_2, u_3 is introduced and an associated Cartesian set ξ, η, ζ, referred to P as origin. A separating line is then defined which divides the plate into two regions, the "central domain" where ordinary rectangular coordinates may be used, and the "boundary domain" in which the above local coordinates must be used. Assuming that the field variations in the direction which starts normal to the plate edge are approximately sinusoidal, it is then possible to match the fields in the two domains across the separating line. The method is also applied to determine the dispersion field due to a hole in the plate conductor. *M. C. Gray* (Murray Hill, N. J.).

Sakurai, Tokio. Approximate evaluation of electromagnetic oscillating modes on plate conductor with slowly curved edge. J. Phys. Soc. Japan **3**, 241–245 (1948).

This is a continuation of the paper reviewed above, and describes an approximate method of evaluating the fields in both domains. The method is applied to the case of a circular plate, and the radius of the plate for resonance in a given oscillating mode is determined. *M. C. Gray.*

Leitner, A., and Spence, R. D. Effect of a circular groundplane on antenna radiation. J. Appl. Phys. **21**, 1001–1006 (1950).

L'antenna è lunga un quarto di lunghezza d'onda, è perpendicolare al disco di base ed ha un estremo al centro di esso. Il problema viene risolto in coordinate sferoidali, mediante costruzione della funzione di Green che soddisfa sul disco di base (perfettamente conduttore) la stessa condizione della forza magnetica. Il campo risulta allora come una serie di funzioni d'onda sferoidali. Gli autori calcolano la distribuzione della corrente sul disco di base, il diagramma di radiazione e la resistenza di radiazione in funzione del raggio del disco. Trovano ottimo accordo con i risultati sperimentali. *G. Toraldo di Francia* (Firenze).

Mirimanov, R. G. The complex radiation resistance of an antenna system in the presence of electromagnetic interaction with another antenna system. Doklady Akad. Nauk SSSR (N.S.) **73**, 1177–1179 (1950). (Russian)

Siano S_1 e S_2 rispettivamente le superficie dei riflettori dell'antenna trasmittente e di quella ricevente, S_0 sia una superficie chiusa di cui fanno parte S_1 e S_2 e che contenga al suo interno le due antenne. Sia $\mathbf{E}_1$, $\mathbf{H}_1$ il campo generato dalla corrente I nell'antenna trasmittente in assenza della ricevente e $\mathbf{E}_2$, $\mathbf{H}_2$ il campo che la stessa corrente nella trasmittente genererebbe all'interno della superficie S_0, supposta conduttrice perfetta. Si ponga $Z_0=(2I_0I_0^*)^{-1}\int_l I^*\mathbf{E}_2\cdot d\mathbf{l}$ essendo l'integrazione estesa all'antenna trasmittente l e avendo indicato con I_0 il valore che I assume in un punto prestabilito di essa. Allora una semplice applicazione del lemma di Lorentz permette di trovare l'impedenza Z dell' antenna trasmittente in presenza della ricevente

$$Z=Z_0+\frac{c}{8\pi I_0I_0^*}\int_{S_0-S_1-S_2}\mathbf{E}_1\times\mathbf{H}_3^*\cdot d\mathbf{S}+\int_\Sigma(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)\cdot d\mathbf{S}.$$

Qui Σ rappresenta una superficie interna a S_0, che racchiude l'antenna trasmittente. *G. Toraldo di Francia* (Firenze).

***de Broglie, Louis. Problèmes de propagations guidées des ondes électromagnétiques.** Gauthier-Villars, Paris, 1951. vii+118 pp. 1,100 francs.

Reprint with minor changes of the 1st edition [1941; these Rev. **8**, 183].

Wagner, Karl Willy. Propagazione di onde elettromagnetiche in conduttori cavi e in cilindri di dielettrico. Ricerca Sci. **20**, 1394–1416 (1950).

Expository lecture.

Cunliffe, A., and Mathias, L. E. S. Some perturbation effects in cavity resonators. Proc. Inst. Elec. Engrs. Part III. **97**, 367–376 (1950).

The authors first briefly summarize the properties of a right-circular cylindrical cavity, and then use a perturbation method to determine the effect of a small change in the cavity volume. It is found that, when the cavity is operating in the E_{010} mode (or any mode whose operating frequency is well separated from all other resonant frequencies), small volume changes have no appreciable effect on the field configuration and merely introduce a small change in the resonant frequency. On the other hand, in the degenerate case of an $H_{011}-E_{111}$ cavity (when the same resonant frequency is associated with three distinct field configurations) a small change in volume may introduce coupling between the fields. Experimentally, the change in volume is obtained by three distinct methods, first by using a plunger attachment, then by tilting the end plates of the cavity and finally by replacing one plane end plate with a conical end plate. In each case the degeneracy is removed, and there are three distinct resonant frequencies, but the first and third deformations have little effect on the field configurations while the second introduces appreciable coupling.

M. C. Gray (Murray Hill, N. J.).

Rinow, Willi. Über eine Anwendung der Störungsrechnung auf das Problem der gekrümmten Leitung. Math. Nachr. **3**, 176–192 (1950).

A perturbation method is used to determine the field in a curved waveguide of arbitrary crossection corresponding to a known mode of propagation in a straight guide of the same crossection. It is assumed that the longitudinal propagation constant and the field components can be expanded in series of inverse powers of the radius of curvature, and it is found that the coefficients of the series satisfy a set of nonhomogeneous differential equations of the second order. The solution of these equations by a method of successive approximations is outlined, and the general theory is then applied to the problem of two straight guides of equal crossection joined by a curved section. The joints are such that the field is continuous across each joint, a field of given mode is applied at the beginning of the input section and the end of the output section is closed through its characteristic impedance. Finally the case of a curved coaxial line is discussed, and explicit formulas are obtained for the first perturbation coefficients corresponding to the dominant TEM wave in the straight coaxial. *M. C. Gray.*

Macfarlane, G. G., and Woodward, A. M. Small-signal theory of wave propagation in a uniform electron beam. Proc. Inst. Elec. Engrs. Part III. **97**, 322–328 (1950).

The authors discuss the use of an electron beam as a waveguide for TM waves. An ideal plane uniform beam is assumed, in which all the electrons move in the z-direction with the same mean velocity V_0 and the beam extends from $y=-a$ to $y=+a$. As long as the kinetic energy of the electrons is large compared with the energy of the wave motion the traveling wave field is approximately linear, and can be expressed in terms of an axial vector potential, $A=(B\cos pa+C\sin pa)\exp i(\omega t-hz)$. The mode numbers p are determined by matching the transverse admittances at the boundaries $y=\pm a$ for the fields inside and outside the beam. Three special cases are considered, namely, the beam in free space, the beam between perfectly conducting parallel planes, and the beam between two reactive sheets which can support slow TM waves. In each case it is shown that there are four waves coexisting in the beam-guide system for each value of p. In general two of these are of the space-charge type, of constant amplitude but different velocity, and two are of the field wave type; above a certain frequency these may be only a slight modification of the forward and reverse waves in an empty guide, while at lower frequencies they may be very fast reverse waves, one amplifying and one attenuating with distance. The different types of wave that may be present in each of the three systems are carefully analyzed, and conditions indicated which would lead to the production of an amplifying wave.

M. C. Gray (Murray Hill, N. J.).

Durand, Émile. Potentiel et champ d'un type particulier de lentille cylindrique. C. R. Acad. Sci. Paris **232**, 314–316 (1951).

Buffery, G. H. A contribution to the algebra of relay and switch contacts. Proc. Inst. Elec. Engrs. Part I. **97**, 357–363 (1950).

The author applies Boolean algebra to switching problems using a notation which is the dual of the notation of C. E. Shannon [Trans. Amer. Inst. Elec. Engrs. **57**, 713–723 (1938)]. *E. N. Gilbert* (Murray Hill, N. J.).

Quantum Mechanics

Castoldi, L. Sulla relazione di indeterminazione. Nuovo Cimento (9) **7**, 961–962 (1950).

Kallmann, H., und Päsler, M. Berechnung der Matrixelemente des H-Atoms mittels der Laplace-Transformation. (Weitere Mitteilung zu: "Eine neue Darstellungsmethode wellenmechanischer Probleme"). Z. Physik 128, 347–365 (1950).

In a series of papers [cf., e.g., same Z. 126, 734–748 (1949); these Rev. 11, 298], the authors have applied the Laplace transform to a variety of quantum mechanical problems. In this paper, they show for the case of the hydrogen atom that the matrix elements $M_n = \int R_{nl} r^n R_{n'l'}\, d\tau$, where R_{nl} give the radial dependence of the wave function, may be expressed in terms of the Laplace transforms of these functions. This is accomplished via the rules $\mathfrak{L}(r^n F(r)) = (-)^n d^n f/dp^n$, where

$$\mathfrak{L}(F) = f(p), \quad \mathfrak{L}(e^{-\beta r} F(r\alpha)) = f(p+\beta),$$

and the Faltung theorem:

$$\mathfrak{L}\{F_1(r) F_2(r)\} = (2\pi i)^{-1} \int f_1(p-z) f_2(z)\, dz.$$

In the case under consideration the contour integral may be evaluated by the use of Cauchy's integral theorem.

H. Feshbach (Cambridge, Mass.).

Mokul'skiĭ, M. A. The relation between the classical and quantum equations of a self-consistent field. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 20, 688–692 (1950). (Russian)

In a paper on the vibrational properties of the classical electron gas Vlasov [same Žurnal 8, 291–318 (1938)] derived an integral equation for the classical distribution function. This equation resembles Hartree's integral equation for the self-consistent field. The author shows how Vlasov's equation can be derived from Hartree's equation by letting the quantum of action h approach zero.

E. Gora (Providence, R. I.).

el Nadi, M. Dirac's equation in a Riemann space with an asymmetrical metric. Physical Rev. (2) 81, 159–160 (1951).

Roberts, K. V. Remarks on relativistic quantum mechanics. Physical Rev. (2) 81, 158–159 (1951).

Koba, Ziro. Note on a Lorentz-invariant integration in the quantum field theory. Progress Theoret. Physics 5, 696–717 (1950).

The author (i) discusses the meaning of the integrability condition for H, the Hamiltonian density, (ii) gives a simple rule for obtaining H from the Lagrangian density L, (iii) introduces a new Lorentz-invariant time ordering operation P^*, (iv) uses P^* to give a direct definition of $U(\sigma_F, \sigma_I)$ in terms of L, (v) derives rules for evaluating the S-matrix showing that it is independent of surface dependent terms and thus proving the equivalence of the Tomonaga-Schwinger theory and Feynman's formalism, and (vi) briefly discusses an example of nonlocal interaction.

The integrability condition is a consequence of requiring that the transformation operator $U(\sigma_F, \sigma_I)$ from an initial surface σ_I to a final surface σ_F be independent, to within a phase factor, of the choice of the intervening family of spacelike surfaces. Kanesawa and the author have previously given a method of determining the Hamiltonian density $H(\mathbf{x}, n(\mathbf{x}))$, where $n(\mathbf{x})$ is the normal at $\mathbf{x}$ to the surface $\sigma(\mathbf{x})$ [Kanesawa and Koba, same journal 4, 297–311 (1949); these Rev. 11, 765]. The method was complicated but had the advantage of providing the interaction Hamiltonian even for a theory which could not be expressed in canonical form. Now the author gives the following "empirical" formula for H in terms of L: $H(\mathbf{x}, n(\mathbf{x})) = \sum_{m=0}^{\infty} H_m(\mathbf{x}, n(\mathbf{x}))$, where

$$H_m = -(-i/2h)^m \times \int^{\sigma(\mathbf{x})} d^4\mathbf{x}_1 \cdots \int^{\sigma(\mathbf{x}_{m-1})} d^4\mathbf{x}_m [L(\mathbf{x}), L(\mathbf{x}_1), \cdots, L(\mathbf{x}_m)]$$

with the generalized commutator defined by induction so that, for example, $[[[A, B], C], D] = [A, B, C, D]$. The author shows that H, so defined, satisfies the integrability condition. The main lacuna in the paper is that no general proof of the above formula for H is given. It is empirical in the sense that it "has been verified in several cases investigated so far," though which cases is not explicitly stated. The author points out that Dyson's time-ordering symbol P is not Lorentz-invariant when it operates on noncommuting observables at events differing by a space-like interval and he therefore defines P^* which can roughly be described as a symmetrization of P. For example, if $\mathbf{x}_1 - \mathbf{x}_2$ is space-like but all other $\mathbf{x}_i - \mathbf{x}_j$ are time-like, then

$$P^*(A_1(\mathbf{x}_1) A_2(\mathbf{x}_2) A_3(\mathbf{x}_3) \cdots) = \tfrac{1}{2}(A_1(\mathbf{x}_1) A_2(\mathbf{x}_2) + A_2(\mathbf{x}_2) A_1(\mathbf{x}_1)) P(A_3(\mathbf{x}_3) A_4(\mathbf{x}_4) \cdots).$$

The author is then able to define $U(\sigma_F, \sigma_I)$ explicitly in terms of $L(\mathbf{x})$ as

$$U(\sigma_F, \sigma_I) = \sum_0^{\infty} \frac{1}{m!} \left(\frac{i}{h}\right)^m \times \int_{\sigma_I}^{\sigma_F} d^4\mathbf{x}_1 \cdots \int_{\sigma_I}^{\sigma_F} d^4\mathbf{x}_m P^*(L(\mathbf{x}_1) L(\mathbf{x}_2) \cdots L(\mathbf{x}_m))$$

and show that it satisfies the Tomonaga-Schwinger equation when H is defined as above. The author discusses the evaluation of $S = U(\infty, -\infty)$ in the case of a neutral scalar field showing that the use of P^* rather than P automatically eliminates surface-dependent terms. Finally, an example of an interaction between radiation and an electron is given which can be interpreted as nonlocal and which gives rise to finite self-energy for the electron.

A. J. Coleman.

Bogolyubov, N. N., Bonč-Bruevič, V. L., and Medvedev, B. V. On the invariant construction of a quantum theory of fields. Doklady Akad. Nauk SSSR (N.S.) 74, 681–684 (1950). (Russian)

Dirac [Rev. Modern Physics 21, 392–399 (1949); these Rev. 11, 409] has investigated the most general possible form of a classical relativistic dynamics describing a system of localizable fields. In this paper the corresponding general form is determined for a theory of relativistic fields quantized according to Bose statistics, but dropping the condition of localizability. The formulae defining such a general theory are complicated and involve many arbitrary functions. This was to be expected, since all kinds of nonlocal interactions must be included as special cases of the general formulae.

F. J. Dyson (Birmingham).

de Broglie, Louis. Sur une forme nouvelle de la théorie du "champ soustractif." J. Phys. Radium (8) 11, 481–489 (1950).

This paper is a new exposition of a theory of the author in which charged particles interact not only with the Maxwell field but also with one or more additional vector fields of nonzero rest-mass. The additional vector fields are "subtractive" in that they give contributions to the particle self-energy which cancel the divergent contribution from the Maxwell field. [For earlier accounts of this theory see

C. R. Acad. Sci. Paris **229**, 157–161, 269–271, 401–404, 640–643 (1949); these Rev. **11**, 302]. *F. J. Dyson.*

Visconti, Antoine. Sur la théorie du champ soustractif de Louis de Broglie et la fonction d'interaction de R. P. Feynman. C. R. Acad. Sci. Paris **232**, 217–219 (1951).

Balseiro, José A. Transformation of configurations and fluctuations of stationary quantum fields. Revista Unión Mat. Argentina **14**, 406–424 (1950). (Spanish. English summary)

McConnell, J. Diverging integrals in the self-charge problem. Physical Rev. (2) **81**, 275 (1951).

Roberts, K. V. On the quantum theory of the elementary particles. I. Introduction and classical field dynamics. Proc. Roy. Soc. London. Ser. A. **204**, 123–144 (1950).

This is the first of a promised series of papers on the formalism of the quantum theory of fields. The classical field dynamics of Weiss [Proc. Roy. Soc. London. Ser. A. **169**, 102–119 (1938)] is extended to include canonical transformations. It is shown that there is a transformation by which a classical analog of the interaction representation of Tomonaga and Schwinger is obtained. Direct quantization in this representation is obtained by considering some variations to take place in a Heisenberg representation and others in a Schrödinger representation. General rules for the construction of the interaction Hamiltonian are obtained from the classical theory. The consistency of the resultant equations is guaranteed by their derivation from a Lagrangian. *K. M. Case* (Ann Arbor, Mich.).

Darling, B. T. The irreducible volume character of events. I. A theory of the elementary particles and of fundamental length. Physical Rev. (2) **80**, 460–466 (1950).

A "fundamental length" is introduced into the theory of elementary particles by means of the postulate that the dynamical laws must be expressed by means of finite difference equations. In this paper the detailed application to spin $\frac{1}{2}$ particles is considered. The Dirac equation is generalized to be a self adjoint finite difference equation. To obtain relativistic invariance this is averaged over the four-dimensional orthogonal group. By considering plane wave solutions of the resultant integral equations it is found that these describe a sequence of particles of different rest-masses. With appropriate choice of the fundamental length the lowest mass agrees with that of the electron, the second with the μ-meson, and the thirteenth with the proton. In addition, there are an infinite number of unidentified particles. For neutral spin $\frac{1}{2}$ particles two versions of the theory are possible. In one there is only one particle, with mass zero. In the other there are also an infinite number of other neutral particles. In neither theory is there a neutral particle of mass intermediate between that of the neutrino and the μ-meson. *K. M. Case* (Ann Arbor, Mich.).

Tanikawa, Yasutaka. A new interpretation of the negative energy Bose field. Progress Theoret. Physics **5**, 692–695 (1950).

The hole theory of negative energy electrons makes use of the fact that electrons obey the exclusion principle. This is not true of bosons so that negative energy bosons have not hitherto been seriously considered. However, the author contemplates the possibility that boson-fields, describable by systems of bosons, do satisfy Fermi-Dirac statistics. In other words he assumes Fermi-Dirac statistics for an ensemble whose elements are not bosons but systems of bosons. *A. J. Coleman* (Toronto, Ont.).

Awano, Tamotsu. On the negative energy photon hypothesis. J. Phys. Soc. Japan **5**, 316–321 (1950).

The quantum electrodynamics proposed by Dirac [Proc. Roy. Soc. London. Ser. A. **180**, 1–40 (1942); these Rev. **5**, 277] using an indefinite metric in Hilbert space is set up in the framework of the covariant formulation of Schwinger [Physical Rev. (2) **74**, 1439–1461 (1948); **75**, 651–679 (1949); these Rev. **10**, 345, 663] and Tomonaga [Progress Theoret. Physics **1**, 27–42 (1946); these Rev. **10**, 226]. Transition is made to the interaction representation. The contact transformation eliminating the first order interaction term is made and the resulting interaction term is separated into portions corresponding to different numbers of particles. The electron self energy is calculated to second order and found to be divergent. The λ limiting process which was used by Dirac along with the indefinite metric to achieve convergence is shown to be inapplicable since the transition to an interaction representation cannot be made. *K. M. Case* (Ann Arbor, Mich.).

Bleuler, K. Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen. Helvetica Phys. Acta **23**, 567–586 (1950).

A method due to Gupta [Proc. Phys. Soc. Sect. A. **63**, 681–691 (1950); these Rev. **12**, 67] is followed to remove the difficulties of normalization and vacuum definition occasioned by the Lorentz condition. The fourth component of the electromagnetic potential is quantized using an indefinite metric, while the Lorentz condition is simultaneously modified so as to apply to positive frequencies only. Lorentz invariance is proved. It is shown that the Maxwell equations hold for expectation values. The lack of uniqueness implied by the indefinite metric is found to be just that due to the possibility of gauge transformations. Complete equivalence with the conventional theory is found after the scalar and longitudinal photons have been eliminated. Starting from a vacuum definition in terms of transverse photons only it is shown by means of a gauge transformation that the symmetrical vacuum definition is obtained with the addition of a term previously found by Coester and Jauch [Physical Rev. (2) **78**, 149–157 (1950); these Rev. **11**, 762]. This additional term does not contribute for processes taking place in an infinite time interval. *K. M. Case.*

Fermi, Enrico. High energy nuclear events. Progress Theoret. Physics **5**, 570–583 (1950).

As an alternative to the conventional perturbation theory a statistical method for computing high energy collisions of protons with multiple production of particles is presented. It is assumed that when high energy nucleons collide, a great amount of energy is released in a small region surrounding the pair. Due to the strong interaction with the π meson field it is expected that this energy will be distributed among the various degrees of freedom according to statistical laws with the restriction that the conservation laws be fulfilled. The relative probabilities of the various processes that can result are then simply given by their statistical weights. As applications, single and multiple meson production cross-sections are obtained. The results are in remarkably good agreement with experiment. For extremely high energies where nucleon pair production may be expected to be significant, a thermodynamic method is

given. Stefan's law of black body radiation is appropriately modified to account for the different statistics and statistical weights to be associated with mesons and nucleons as compared to photons. Extremely simple formulae for the numbers of particles expected to be emitted in very high energy collisions are obtained. *K. M. Case.*

Araki, Gentaro. Production of scalar and pseudoscalar mesons by photons. Progress Theoret. Physics **5**, 507–518 (1950).

Differential and total cross-sections for photomeson production in the scalar and pseudoscalar meson theories are obtained. The processes $\gamma+P\rightarrow N+\pi^+$; $\gamma+N\rightarrow P+\pi^-$; and $\gamma+P\rightarrow P+\pi^0$ are considered. Before detailed calculation is begun the Hamiltonian is simplified by means of a sequence of contact transformations. Conventional perturbation theory is then used in the lowest nonvanishing approximation. In agreement with the work of Brueckner and Goldberger [Physical Rev. (2) **76**, 1725 (1949)], it is found that the negative to positive and neutral to positive ratios are particularly simple. Curves for these ratios and for the total cross-sections as functions of photon energy are included. *K. M. Case* (Ann Arbor, Mich.).

Wentzel, Gregor. Nuclear saturation phenomena deducible from pair theories. Progress Theoret. Physics **5**, 584–599 (1950).

In the static approximation, pair theories of nuclear forces are somewhat easier to treat than the Yukawa theory. The author uses pair theories to see what features of nuclear saturation phenomena can be found from a field theory. Previously obtained solutions [Helvetica Phys. Acta **15**, 111–126 (1942); A. Houriet, ibid. **16**, 529–550 (1943); these Rev. **5**, 279] are extended to the case where several nucleons are present in a region small compared to the nuclear sources function. Strong coupling is applied to those cases where no rigorous solution is available and to obtain the interaction energy of a cluster. Nuclear models ranging from single nucleon to α-particle models are suggested by the results of different types of theories. It is found that the population of a small cell cannot be increased beyond a certain maximum occupation number n_s except at high cost in energy. The interaction assumed determines n_s. It is also shown that for some pair theories the mutual potential energy of two small cells which are separated by at least a source radius and contain n_1 and n_2 nucleons, respectively, approaches a constant value as n_1 and n_2 increase.
K. M. Case (Ann Arbor, Mich.).

Maue, A.-W. Magnetische Momente von Elementarteilchen mit höherem Spin. Z. Physik **128**, 378–386 (1950).

Hönl and Boerner [Z. Naturforschung **5a**, 353–366 (1950); these Rev. **12**, 380] have obtained the irreducible wave equations resulting from the fusion of three Dirac particles. These are a 4-dimensional representation equivalent to the Dirac equation and a 16- and 20-dimensional representation which both describe particles with spin $\frac{1}{2}$ and $\frac{3}{2}$ and masses in the ratio 3:1. In this paper the magnetic moments of these higher spin particles are calculated. Explicit representations are given for the matrices involved. The magnetic moments are found from a second order perturbation calculation of the energy of a stationary particle in a uniform magnetic field. In some cases degeneracy requires the unperturbed wave function to be a combination of states of different total space. This makes the determination of a "g" value ambiguous. The resulting magnetic moments cannot be identified with those of any known particles. *K. M. Case* (Ann Arbor, Mich.).

Hepner, W. A. A canonical transformation in the theory of particles of arbitrary spin. Physical Rev. (2) **81**, 290–291 (1951).

Kwal, Bernard. Les potentiels généralisés en théorie du corpuscle de spin 1 en présence des sources, et la difficulté fondamentale de la théorie du méson. C. R. Acad. Sci. Paris **232**, 37–38 (1951).

Vrkljan, V. S. On meson spin. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 203–208 (1949). (Serbo-Croatian)

This paper presents several ideas from earlier theoretical work on mesons, including that of H. Yukawa [Proc. Phys.-Math. Soc. Japan **17**, 48–57 (1935)] and de Broglie [Théorie générale des particules à spin . . . , Gauthier-Villars, Paris, 1943]. *N. Rosen* (Chapel Hill, N. C.).

Avak'yanc, G. M. The interaction of γ-rays with a deuteron. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 669–672 (1950). (Russian)

The author calculates the effective crossection for the disintegration of a deuteron by γ-rays, taking into account the exchange current between the proton and the neutron [R. G. Sachs, Physical Rev. (2) **74**, 433–441 (1948)]. He finds that inclusion of the effect of the exchange current leads to a considerable change in the size of the cross-section. For example, with a square-well neutron-proton interaction potential having a depth of 20 mev. and a radius of 2.8×10^{-13} cm., and with the force assumed to be half exchange, the effect of the exchange current is found to increase the disintegration crosssection for 40 mev. photons by about 140%. *N. Rosen* (Chapel Hill, N. C.).

***Rosenfeld, L. Theory of Electrons.** Selected Topics in Modern Physics. I. North-Holland Publishing Co., Amsterdam; Interscience Publishers, Inc., New York, N. Y., 1951. xv+119 pp. (1 plate). $2.25.

Thermodynamics, Statistical Mechanics

Crawford, F. H. Thermodynamic relations in n-variable systems in Jacobian form. I. General theory and application to unrestricted systems. Proc. Amer. Acad. Arts Sci. **78**, 165–184 (1950).

Principal result: Any thermodynamic derivative can be expressed as the ratio of two Jacobians involving not more than $\frac{1}{2}n(n+1)$ members of a standard set of derivatives. The computational procedure is illustrated for a thin rod and a triclinic crystal subjected to strain.
C. C. Torrance (Annapolis, Md.).

Niehrs, Heinz. Herleitung einer Mengenfunktion aus einer "assoziativen" Funktion und die Begriffsbildung physikalischer Grössen. Z. Physik **127**, 187–193 (1950).

This is an expository contribution to elementary phenomenological thermodynamics. The notion of quantity of heat (mass, electric charge, etc.) can be based on the simple concept of "mixing": an associative commutative continuous binary operation. By means of a simple functional equation, the familiar properties of heat and temperature are derived. *B. O. Koopman* (New York, N. Y.)

Surinov, Yu. A. Concerning some fundamental equations of the theory of heat radiation. Doklady Akad. Nauk SSSR (N.S.) **72**, 469–472 (1950). (Russian)

Let $B(M, s, \tau)$ be the intensity of radiation at point M, for time τ, and in the direction s. The author defines the vector spherical radiation $\mathbf{E}_{4\pi}(M, \tau) = \int_{(4\pi)} B\, d\omega$, where $d\omega = r_1\, d\omega(M, s)$ is a vector element of the solid angle ω, and r_1 is a unit vector in the direction s. In the stationary case, which the author considers almost exclusively, he shows that $\operatorname{div} \mathbf{E}_{4\pi}(M) = 0$, and continues with discussions of the hemispherical vector radiation, $\mathbf{E}_{2\pi}(M, n) = \int_{(2\pi)} B\, d\omega$, and the projections, $(\mathbf{E}_{4\pi}, \mathbf{n}_1)$ and $(\mathbf{E}_{2\pi}, \mathbf{n}_1)$, on lines from M normal to a boundary surface F. He arrives at an integral equation for the radiation at F, given the temperature field, the configuration of the system, and the optical constants.

R. E. Gaskell (Ames, Iowa).

Surinov, Yu. A. On the solution of the problem of radiative transfer in systems of grey bodies. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 1345–1376 (1950). (Russian)

The author states that this article is a continuation of two of his earlier works [see the preceding review; same Izvestiya Otd. Tehn. Nauk **1948**, 981–1002] on the development of elementary algebraic methods for the solution of the problem on the radiation exchange in a system of grey bodies. Systems of convex and of nonconvex radiating bodies are considered. *H. P. Thielman* (Ames, Iowa).

Predvoditelev, A. S. On the statistical ensemble of material points with variable mass. Vestnik Moskov. Univ. **3**, no. 6, 41–49 (1948). (Russian)

This paper extends the author's earlier work [same Vestnik **1947**, no. 7, 23–42; these Rev. **11**, 633] to ensembles of relativistic particles. Using the procedure developed in the paper cited above the author starts with a classical relativistic Lagrangian and arrives at Schrödinger's relativistic wave equation, at Dirac's relativistic equation, and at generalizations of both which have not yet found physical applications. The author points out that a relation exists between the operator method of wave-mechanics, and the method of Riemann-Hugoniot-Hadamard, which he hopes will shed light on the operator method and will suggest its correct generalizations. *G. M. Volkoff.*

Raševskiĭ, P. K. The statistics of Bose-Einstein and Fermi-Dirac from the tensor point of view. Trudy Sem. Vektor. Tenzor. Analizu **7**, 362–380 (1949). (Russian)

In the usual presentations of the quantum theory of an ensemble of identical particles the tensor point of view is commonly not preserved throughout. The author claims that if the tensor point of view is retained throughout the calculations are both simpler, and more easily interpreted. Mathematical sections on properties of tensors are followed by explicit applications of tensors to quantum theory.

G. M. Volkoff (Vancouver, B. C.).

Isenberg, Irvin. The virial theorem and the variation principle. Physical Rev. (2) **79**, 737 (1950).

The author points out that since the existence of a virial theorem in quantum mechanics depends only on the existence of an energy variation principle, the former continues to apply even when the true wave function is replaced by certain approximations obtained by variational methods. [See ibid. **80**, 917 (1950).]

B. O. Koopman (New York, N. Y.).

Cox, R. T. The statistical method of Gibbs in irreversible change. Rev. Modern Physics **22**, 238–248 (1950).

The author obtains a number of properties of systems not in equilibrium by using the methods of statistical mechanics. He considers two "loosely coupled" systems in various states, the second being regarded as the environment of the first. He discusses approach to equilibrium, viscous forces, recovery from a sudden disturbance of equilibrium, changes in an isolated system, minimal principles in isolated systems. Although there are many equations in the paper, the methods of obtaining them and of passing from one to another are of the nonmathematical, heuristic type usual in physical works on statistical mechanics. *C. Truesdell.*

Koga, Toyoki. On states not in thermal equilibrium (induction of the equation of motion of gas). J. Phys. Soc. Japan **4**, 34–38 (1949).

The author proposes complicated equations for stress and heat flux in a rarified gas, based on heuristic kinetic theory arguments. His method is apparently to set up promising expressions containing arbitrary constants and then to adjust these constants so that the equations of transfer are approximately satisfied. He proposes boundary conditions also. *C. Truesdell* (Bloomington, Ind.).

Mayer, J. E. Distribution functions and integral equation methods. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 209–226 (1949).

A discussion of methods, involving Fredholm integral equations, for evaluating certain expressions which occur in statistical mechanics of systems which may contain different phases. The usual restrictive assumptions are replaced by much more general ones. *B. O. Koopman.*

Husimi, Kodi. Note on Mayers' theory of cluster integrals. J. Chem. Phys. **18**, 682–684 (1950).

The cluster integrals in Mayer's condensation theory, namely I_n, where

$$N!I_n = \int^{(N)} \cdots \int W(1, 2, \cdots, N)\, d1\, d2 \cdots dN,$$

$W(1, 2, \cdots, N) = \exp[-V(1, 2, \cdots, N)/kT]$ (the integers $1, 2, \cdots, N$ denoting respective coordinates of the N similar molecules and V denoting the energy) are evaluated by an expansion of W into sums of products of certain symmetric types. The problem of evaluation is thus reduced to the combinatorial problem of counting generalized Cayley "trees." This is solved by means of generating functions (generalizing a method of G. Bol). The results have wide applicability to gas statistical mechanics.

B. O. Koopman (New York, N. Y.).

Mutô, Yosio. Notes on statistical thermodynamics of solutions and gaseous mixtures. I, II, III. J. Phys. Soc. Japan **4**, 233–237, 238–242, 242–245 (1949).

These papers are concerned with the application of the Mayer theory of imperfect gases to the development of a formalism for finding the thermodynamic properties of gas mixtures and multicomponent solutions [see also J. E. Mayer, J. Phys. Chem. **43**, 71–95 (1939); K. Fuchs, Proc. Roy. Soc. London. Ser. A. **179**, 408–432 (1942); these Rev. **4**, 30; McMillan and Mayer, J. Chem. Phys. **13**, 276–305 (1945)]. In this theory thermodynamic properties are expressed as power series in density or in concentration, the

coefficient kth power depending on integrals over clusters of k-particles. The author introduces a set of "semi-reducible integrals" over clusters containing fixed numbers of molecules of each of the species occurring in the system. He discusses thermodynamic properties in terms of these integrals as well as the "irreducible integrals" defined by Mayer. No special problems are solved in the paper by using the formalism there developed. *E. W. Montroll.*

Köhler, Hilding. On the problem of condensation in the atmosphere. Nova Acta Soc. Sci. Upsaliensis (4) **14**, no. 9, 76 pp. (1950).

Montroll, E. W. Continuum models of cooperative phenomenon. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 265–278 (1949).

Mathematically, the simplest problems in cooperative phenomena are concerned with the evaluation of a sum in the form $P=\sum_{\sigma_j} \exp\left(\sum_{(i,j)} -H\sigma_i\sigma_j\right)$, where the variables σ_j have the values $+1$ or -1 (i.e., $\sigma_j^2=1$) and the pairs i, j are taken according to a geometric rule (for instance, the indices may indicate points on a plane or cubical lattice and the pairs are neighboring pairs). The author proposes to approximate P and similar expressions by replacing the conditions $\sigma_j^2=1$ by a set of conditions $\sigma^2_{s_1}+\cdots+\sigma^2_{s_k}=k_l$ and the outer summation by integration over these spheres R_l. If one has m spheres R_l, the appropriately normalized expression corresponding to P is denoted by Q_m. The author then shows how one can evaluate these Q_m, using techniques from the theory of probability and integral equations, provided one can solve a certain characteristic value problem. For $m=1$, i.e., for one sphere $\sum\sigma_j^2=N$, this process is carried out in detail for a linear, square, and cubical lattice. A phase transition occurs in the three but not in the two-dimensional Q_1. The author also refers to certain unpublished work of Berlin and Kac who successfully treated Q_1 in the three-dimensional case and showed the existence of a temperature interval for long range order. There is appended a remark of L. Onsager pointing out a difficulty the above approximating process must overcome. *F. J. Murray.*

Bowers, Wayne A., and Rosenstock, Herbert B. On the vibrational spectra of crystals. J. Chem. Phys. **18**, 1056–1062 (1950).

This is a calculation of the exact distribution of frequencies of normal modes of vibration of a model of a two-dimensional array of $N\times N$ atoms. The atomic vibrations are assumed to be in the direction normal to the plane of the atoms (rather than in the plane or along the lines connecting atoms as would occur in real crystals). It is shown that the frequency distribution can be expressed in closed form as a complete elliptic integral. There is a logarithmic infinity in this distribution. The results are very similar to those obtained by the reviewer [same J. **15**, 575–591 (1947)] for a lattice with vibrations in the plane of the atoms.

The exact frequency distribution and specific heat for the model discussed in this paper are compared with approximate results obtained by methods proposed by W. V. Houston [Rev. Modern Physics **20**, 161–165 (1948)] and the reviewer [same J. **11**, 481–495 (1943)] for real crystals. Houston's approximation yields very good results in the low-frequency range, but leads to spurious infinities. T. Nakamura has obtained similar effects in a lattice with vibrations in the plane of the atoms [see the following review]. The reviewer's approximation leads to a distribution which agrees quite well with the exact one at all frequencies except those in the neighborhood of the logarithmic infinity. The reviewer's approximation leads to better specific heats in the high-temperature region, while that of Houston is superior at low temperatures.

The frequency distribution of a model of a three-dimensional crystal whose atoms vibrate in the fourth dimension is plotted. The logarithmic infinities of the two-dimensional case seem to become finite peaks in this model. The authors suggest that the peaks in a real three-dimensional crystal might be finite rather than infinite. *E. Montroll.*

Nakamura, Tutô. Note on the theory of the frequency spectrum of crystalline solid. Progress Theoret. Physics **5**, 213–223 (1950).

W. V. Houston [Rev. Modern Physics **20**, 161–165 (1948)] has developed an approximation method for determining the distribution of the frequencies of normal modes of vibration of an atomic lattice. This distribution is necessary for the computations of the thermodynamic properties of solids. This paper applies the Houston method to the calculation of the frequency distribution in a two-dimensional square lattice and compares the results with the exact distribution function obtained by the reviewer [J. Chem. Phys. **15**, 575–591 (1947)]. The exact distribution has two logarithmic infinities. The approximate one agrees with the exact one at low frequencies and is in fair agreement with it for all frequencies smaller than the low frequency infinity. At higher frequencies the approximation shows several spurious infinities. The approximate distribution function in the neighborhood of an infinity at ν_0 is proportional to $[1-(\nu/\nu_0)^2]^{-1}$ rather than $\log[1-(\nu/\nu_0)]$ as in the exact distribution function. Similar comparisons have been made by S. Huzinaza [in an unavailable paper in Japanese] and by Bowers and Rosenstock [see the preceding review]. *E. Montroll* (College Park, Md).

Nambu, Yôichirô. A note on the eigenvalue problem in crystal statistics. Progress Theoret. Physics **5**, 1–13 (1950).

The author outlines a method of solving the eigenvalue problem which arises in the determination of the partition function for an Ising lattice. The method has been used by other workers on this problem and is essentially the following. The 2^n dimensional operator whose eigenvalues are to be found is written as a product of operators, each of which is the spin representative of a rotation in a $2n$-dimensional space. The eigenvalues of the rotations all may be determined in a simple manner and the original problem is solved using these results. *A. H. Taub* (Urbana, Ill.).

Trefftz, Eleonore. Zur Statistik der Mischkristalle und Ferromagnetica. Z. Physik **127**, 371–380 (1950).

Approximate high and low temperature expressions are obtained for the partition function and internal energy of simple cubic, body-centered cubic, and face-centered cubic crystal lattices with Ising interactions between particles at lattice points. The author uses combinatory methods similar to those of van der Waerden [same Z. **118**, 473–488 (1941)]. A large number of terms are calculated for each of the lattices mentioned above, and approximate Curie temperatures are given. The theoretical specific heats are compared with experimental ones for iron and nickel. As is to be expected, the low and high temperature approximations are not good in the neighborhood of the Curie point. *E. Montroll* (College Park, Md.).